INTERPOLATION OF DEFLECTION OF THE VERTICAL BASED ON GRAVITY GRADIENTS

L. VÖLGYESI
Department of Geodesy
Technical University of Budapest, H-1521 Budapest
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Abstract

Significance of interpolation of deflection of the vertical by means of torsion balance measurements is pointed out, followed by outlining its fundamentals. Thereafter, its practical methods of solution will be presented.

Keywords: deflection of the vertical, torsion balance measurements, gravity gradients.

Introduction

Knowledge of deflection of the vertical is essential in geodesy, relating to positioning data measurable in Earth's real gravity field and those computable in some normal gravity field. At the same time, knowledge of deflections of the vertical offers an important possibility of the detailed geoid determination. For geoid determination, a dense net of values of deflection of the vertical is necessary. Astrogeodetic determination of deflection of the vertical is extremely expensive and tedious, therefore in practice a sparser net of astronomical stations has to be put up with and this astrogeodetic net is interpolated by different methods.
Interpolating the values of deflection of the vertical may be made either by gravimetric interpolation methods involving gravity anomalies, or - with the knowledge of curvature gradients of potential surfaces of the gravity field - by using torsion balance measurements. From among the two methods, practical applicability of the former is rather restricted, adequate accuracy being conditioned by the availability of detailed gravity data around the point to be determined at a distance of min. 2000 km. Besides, the gravimetric interpolation method is excessively computation-intensive and difficult to be programmed.

All these urge to consider the interpolation of deflection of the vertical based on torsion balance measurements. Under Hungarian conditions, in addition to gradient values W_{zx} and W_{zy} , also curvature data W_{xy} and $W_{\Delta} = W_{yy} - W_{xx}$ are available with great precision. Since earlier torsion balance measurements were made mainly for geophysical prospecting, mostly only gravity gradients have been processed. Up to now, gravity curvature values essential in geodesy - rather promising for detailed determination of deflections of the vertical - have been left unprocessed.

Loránd Eötvös was the first to point out that interpolation of deflection of the vertical is possible from torsion balance measurements, and made also relevant trial computations (EÖTVÖS 1906, 1909; SELÉNYI 1953). The method of Eötvös was further developed in a simplified form by János Renner (RENNER 1952, 1956, 1957), without having an opportunity to safely check the computations. Besides the two Hungarian scientists above, only two members from the staff of the Columbus University USA: J. Badekas and I. Mueller (BADEKAS - MUELLER 1967), as well as U.Heineke in Hannover (HEINEKE 1978) had been concerned with the subject, - but even their works had still much to be cleared.

After outlining the fundamentals of interpolation of deflection of the vertical relying on torsion balance measurements, possible practical computation methods will be presented.

Actually, this seems to be the most economical method for interpolating deflections of the vertical, thereby for precise geoid determinations.

1. Fundamentals of the interpolation method

Let us consider distribution of deflections of the vertical in a small area of Earth's surface where torsion balance measurements are available.

Let computations be referred to a Cartesian system, having an arbitrary point P_0 within the examined area as origin. Let $+x$ and $+y$ be the axes of the system point to the north and to the east, respectively, and let axis z coincide with vertical direction at P_0 so that its positive branch points downwards. Although these directions vary from point to point, at any point of a moderate area - of a size at most $0.5^o \times 0.5^o$ the same coordinate directions may be taken to a fair approximation, namely the effect of deviation due to meridian convergence is within the range of reliability of observations (SELÉNYI 1953).

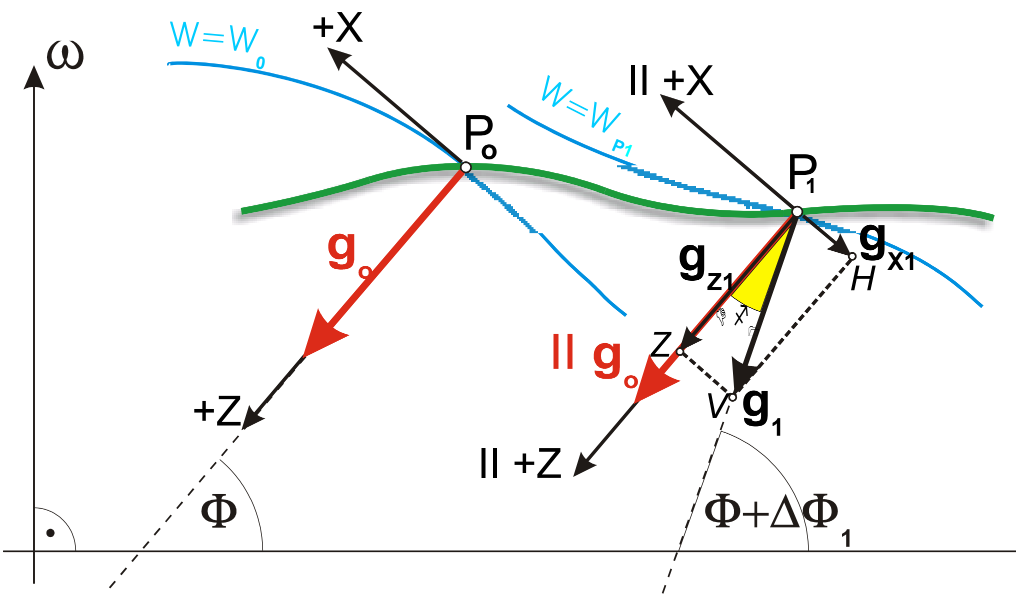

Fig. 1

Thereby, direction z at any point P_i of the concerned area is parallel to the z-axis through point P_0, and the direction x_i to the tangent of astronomical meridian through point P_0, as illustrated by the arbitrary point P_i (actually $i=1$) in *Fig. 1*. The z-axis at point P_1 being parallel to the vertical at origin P_0, presumably, direction of vector g_1 at point P_1 does not coincide with direction z. In *Fig.1*., vector $\overline{P_1V}$ is, in fact, projection of vector g_1 on plane xz, while vector $\overline{P_1H}$ is projection of component g_{x1} of vector

g_1 on the same plane. (There are negligible deviations between vectors $\overline{P_1V}$ and g_1, as well as $\overline{P_1H}$ and g_{x1}).

Be Φ the astronomical latitude of point P_0, and let $\Delta\Phi_1$ symbolize the angle between direction P_1V and z at point P_1, so the astronomical latitude of point P_1 is:

$$\Phi_1 = \Phi + \Delta\Phi_1$$

While, according to Fig.1:

$$-g_{x1} = g_1 \sin\Delta\Phi_1$$

it is to be written, for a small angle $\Delta\Phi_1$, as:

$$\Delta\Phi_1 = -\frac{g_{x1}}{g_1} \tag{1}$$

The same train of thought leads for the variation of astronomic longitude in plane yz to:

$$\Delta\Lambda_1 \cos\Phi_1 = \frac{g_{y1}}{g_1} \tag{2}$$

Equations (1) and (2) yield components N and E of the angle between geoid normal at points P_0 and P_1. Values $\Delta\Phi_2$ and $\Delta\Lambda_2$ for P_0 and some P_2 may be determined in a similar way. These may be applied for writing differences between P_1 and P_2:

$$(\Delta\Phi_2 - \Delta\Phi_1) = -\frac{1}{\tilde{g}}\left(g_{x2} - g_{x1}\right) = -\frac{1}{\tilde{g}}\left[\left(\frac{\partial W}{\partial x}\right)_2 - \left(\frac{\partial W}{\partial x}\right)_1\right] \tag{3}$$

and

$$(\Delta\Lambda_2 - \Delta\Lambda_1)\cos\tilde{\Phi} = -\frac{1}{\tilde{g}}\left(g_{y2} - g_{y1}\right) = -\frac{1}{\tilde{g}}\left[\left(\frac{\partial W}{\partial y}\right)_2 - \left(\frac{\partial W}{\partial y}\right)_1\right] \tag{4}$$

where W is the potential of Earth's real gravity field, while $\tilde{g}$ and $\tilde{\Phi}$ are the mean values of gravity, and astronomical latitude between points P_1 and P_2. By analogy with (1) and (2), (3) and (4) yield components N and E of the angle included by level surface normal at P_1 and P_2.

By introducing notations $\frac{\partial W}{\partial x} = W_x$ and $\frac{\partial W}{\partial y} = W_y$, Eqs. (3) and (4) may be written as:

$$\Delta\Phi_2 - \Delta\Phi_1 = -\frac{1}{\tilde{g}}\left(W_{x2} - W_{x1}\right) \tag{5}$$

and,

$$\left(\Delta\Lambda_2 - \Delta\Lambda_1\right)\cos\widetilde{\Phi} = -\frac{1}{\widetilde{g}}\left(W_{y2} - W_{y1}\right) \tag{6}$$

respectively.

Level surfaces of the potential of normal gravity field, normal gravity, and directions of normal gravity vectors, in this relation, geodetic latitude and longitude of any point, termed normal geodetic latitude ${}_n\varphi$ and normal geodetic longitude ${}_n\lambda$, may be interpreted on the analogy of the Earth's real gravity field.

Relationships similar to (5) and (6) may be written between the variation of the gravity field direction in normal gravity field, that is, of normal geodetic coordinates ${}_n\varphi$ and ${}_n\lambda$ of points P_1 and P_2, and the derivatives conform to potential of the normal gravity field (normal potential):

$$\Delta_n\varphi_2 - \Delta_n\varphi_1 = -\frac{1}{\widetilde{\gamma}}\left(U_{x2} - U_{x1}\right) \tag{7}$$

and

$$\left(\Delta_n\lambda_2 - \Delta_n\lambda_1\right)\cos\widetilde{\varphi} = -\frac{1}{\widetilde{\gamma}}\left(U_{y2} - U_{y1}\right) \tag{8}$$

where U is the normal potential, and $\widetilde{\gamma}$ is the mean value of normal gravity between points P_1 and P_2.

Inside a limited area of size $0.5^o \times 0.5^o$, approximations $\widetilde{\gamma} = \widetilde{g}$ and $\widetilde{\Phi} = {}_n\widetilde{\varphi} = \widetilde{\varphi}$ are permissible - and so are single values $\widetilde{g}$ and $\widetilde{\varphi}$ valid for all the area rather than between two neighboring points alone (BADEKAS and MUELLER 1967), to be indicated simply by g and φ.

Let us subtract Eqs (5) and (7), as well as (6) and (8) from each other:

$$\left[\left(\Delta\Phi_2 - \Delta_n\varphi_2\right) - \left(\Delta\Phi_1 - \Delta_n\varphi_1\right)\right] g = -\left(W_{x2} - W_{x1}\right) + \left(U_{x2} - U_{x1}\right) \tag{9}$$

$$\left[\left(\Delta\Lambda_2 - \Delta_n\lambda_2\right) - \left(\Delta\Lambda_1 - \Delta_n\lambda_1\right)\right] g\cos\varphi = -\left(W_{y2} - W_{y1}\right) + \left(U_{y2} - U_{y1}\right) \tag{10}$$

By definition, differences (9) and (10) between astronomic and normal geodetic latitudes and longitudes yield differences of components ξ and η of deflection of the vertical between points P_1 and P_2:

$$\left(\xi_2 - \xi_1\right)g = -\left(W_{x2} - W_{x1}\right) + \left(U_{x2} - U_{x1}\right), \tag{11}$$

$$\left(\eta_2 - \eta_1\right)g = -\left(W_{y2} - W_{y1}\right) + \left(U_{y2} - U_{y1}\right). \tag{11}$$

Introducing notations

$$\Delta\xi_{12} = \xi_2 - \xi_1 ,$$
$$\Delta\eta_{12} = \eta_2 - \eta_1$$

and

$$\Delta W = W - U \tag{13}$$

leads to equations:

$$g\Delta\xi_{21} = -\Delta W_{x2} + \Delta W_{x1} \cdot, \quad (14$$
$$g\Delta\eta_{21} = -\Delta W_{y2} + \Delta W_{y1} \ . \quad (15$$

Remind that in classic geodesy, deflection of the vertical is frequently interpreted as:

$$\xi = \Phi - \varphi$$
$$\eta = (\Lambda - \lambda)\cos\varphi$$

where Φ and Λ are astronomic co-ordinates, while φ and λ are geodetic (ellipsoidal) co-ordinates of point.

By physically interpreting the ellipsoid, serving as reference surface, as one level surface of the normal gravity field, then ellipsoidal and normal geodetic co-ordinates are related as:

$$\varphi = {}_n\varphi - \kappa \ , \tag{16}$$
$$\lambda = {}_n\lambda \ , \tag{17}$$

where κ is the difference of directions of the normal gravity field between point P on the earth surface and the ellipsoid surface along the normal plumb line at point P. In (16) and (17), normal plumb line being a plane curve lying in the normal meridian plane of point P has been reckoned with.

For an altitude h of point P over the ellipsoid, applying curvature of the plumb line of normal gravity field:

$$\kappa = h\frac{\beta}{R}\sin 2\varphi \tag{18}$$

where β is the dynamical flattening of normal gravity field, and R is the Earth's radius (MAGNITZKI and BROVAR 1964).

By differentiating (18), it is obvious that in the mentioned $0.5^o \times 0.5^o$ area, variation of κ is practically negligible. Hence, (11) and (12) are also valid for the classical geodetic interpretation of deflection of the vertical.

Thus, in the following, when interpreting of deflection of the vertical it is needless to distinguish between the two conceptions, permitting to use the concept of deflection of the vertical in both interpretations.

Components of deflections of the vertical - more closely, their values multiplied by g, that is, horizontal components - seemed to be determined by first derivatives of the potential. While torsion balance measurements yield second derivatives

$$W_\Delta = \frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2} \quad \text{and} \quad W_{xy} = \frac{\partial^2 W}{\partial x \partial y} \ .$$

Thus, the computation problem is essentially an integration to be solved by approximation.

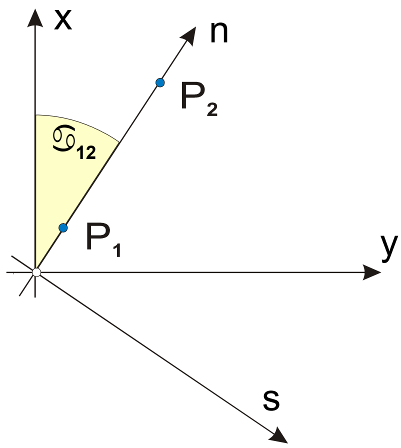

Fig. 2

To this aim, first the co-ordinate transformation in *Fig.2* will be performed, according to matrix equation

$$\begin{bmatrix} n \\ s \end{bmatrix} = \begin{bmatrix} \cos\alpha_{12} & \sin\alpha_{12} \\ -\sin\alpha_{12} & \cos\alpha_{12} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Accordingly:

$$\begin{aligned} W_n &= \frac{\partial W}{\partial n} = \frac{\partial W}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial W}{\partial y}\frac{\partial y}{\partial n} = W_x \cos\alpha_{12} + W_y \sin\alpha_{12} \\ W_s &= \frac{\partial W}{\partial s} = \frac{\partial W}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial W}{\partial y}\frac{\partial y}{\partial s} = -W_x \sin\alpha_{12} + W_y \cos\alpha_{12} \end{aligned} \qquad (19)$$

while second derivatives are:

$$\begin{aligned} \frac{\partial^2 W}{\partial n^2} &= \frac{\partial^2 W}{\partial x^2}\cos^2\alpha_{12} + \frac{\partial^2 W}{\partial y^2}\sin^2\alpha_{12} + \frac{\partial^2 W}{\partial x \partial y}\sin 2\alpha_{12} \\ \frac{\partial^2 W}{\partial s^2} &= \frac{\partial^2 W}{\partial x^2}\sin^2\alpha_{12} + \frac{\partial^2 W}{\partial y^2}\cos^2\alpha_{12} + \frac{\partial^2 W}{\partial x \partial y}\sin 2\alpha_{12} \end{aligned} \qquad (20)$$

and

$$\frac{\partial^2 W}{\partial n \partial s} = \frac{\partial^2 W}{\partial x \partial y}\cos 2\alpha_{12} + \frac{1}{2}\left(\frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2}\right)\sin 2\alpha_{12}$$

This latter $W_{ns} = \dfrac{\partial^2 W}{\partial n \partial s}$ seems to result from torsion balance measurements, with the knowledge of azimuth α_{12} of the direction connecting the two points examined.

Now, by integrating the left-hand side of (20) between limits n_1 and n_2 :

$$\int_{n_1}^{n_2} \frac{\partial^2 W}{\partial n \partial s} dn = \left(\frac{\partial W}{\partial s}\right)_2 - \left(\frac{\partial W}{\partial s}\right)_1 = W_{s2} - W_{s1} \tag{21}$$

If points P_1 and P_2 are close enough to let variation of second derivative W_{ns} be considered as linear, then integral (21) may be computed by trapezoid integral approximation formula:

$$\int_{n_1}^{n_2} \frac{\partial^2 W}{\partial n \partial s} dn = \frac{1}{2}\left[\left(\frac{\partial^2 W}{\partial n \partial s}\right)_1 + \left(\frac{\partial^2 W}{\partial n \partial s}\right)_2\right](n_2 - n_1) = \frac{1}{2}\left[(W_{ns})_1 + (W_{ns})_2\right] n_{12} \tag{22}$$

where $n_{12} = n_2 - n_1$ is the distance between points P_1 and P_2.

On the other hand, by applying transformation (19), integral (21) yields:

$$W_{s2} - W_{s1} = -(W_{x2} - W_{x1})\sin\alpha_{12} + (W_{y2} - W_{y1})\cos\alpha_{12} \quad . \tag{23}$$

The same train of thought yields a similar expression for potential U of normal gravity field:

$$U_{s2} - U_{s1} = -(U_{x2} - U_{x1})\sin\alpha_{12} + (U_{y2} - U_{y1})\cos\alpha_{12} \quad . \tag{24}$$

Subtracting (23) from (24) yields variation $\Delta\Theta_{12}$ of horizontal force component between points P_1 and P_2 in direction n. By taking (23) into consideration, and introducing notation

$$g\Delta\Theta_{12} = G_{12} \tag{25}$$

the following is yielded:

$$G_{12} = (-\Delta W_{x2} + \Delta W_{x1})\sin\alpha_{12} - (-\Delta W_{y2} + \Delta W_{y1})\cos\alpha_{12}$$

after substituting (14) and (15):

$$G_{12} = g\Delta\xi_{21}\sin\alpha_{12} - g\Delta\eta_{21}\cos\alpha_{12}$$

or by introducing notation

$$T_{12} = \frac{G_{12}}{g}$$

equation

$$T_{12} = \Delta\xi_{21}\sin\alpha_{12} - \Delta\eta_{21}\cos\alpha_{12} \tag{26}$$

is yielded.

The left-hand side of (26) may be computed by using (22). When using notation (13):

$$T_{12} = \frac{1}{2}\left[\left(\Delta W_{ns}\right)_1 + \left(\Delta W_{ns}\right)_2\right]\frac{n_{12}}{g} \tag{27}$$

with ΔW_{ns} to be computed from (20):

$$\Delta W_{ns} = \Delta W_{\Delta} \sin 2\alpha_{12} + \Delta W_{xy} \cos 2\alpha_{12} \tag{28}$$

where $\Delta W_{\Delta} = W_{\Delta} - U_{\Delta}$ and $\Delta W_{xy} = W_{xy} - U_{xy}$. Remind that W_{Δ} and W_{xy} are gradients obtainable from torsion balance measurements, while U_{Δ} and U_{xy} are gradients of the normal gravity field, referred to, e.g., the Hayford ellipsoid, in Eötvös units (HEINEKE 1978):

$$U_{\Delta} = 10.26 \cos^2 \varphi \tag{29a}$$
$$U_{xy} = 0 \tag{29b}$$

Now, by substituting (28) into (27):

$$T_{12} = \frac{n_{12}}{4g}\left[\left(\Delta W_{\Delta_1} + \Delta W_{\Delta_2}\right)\sin 2\alpha_{12} + \left(\Delta W_{xy_1} + \Delta W_{xy_2}\right)\cos 2\alpha_{12}\right] \tag{30}$$

which, compared to (26) yields the basic equation wanted, relating the variation of components of deflection of the vertical between two points to gradients from torsion balance measurements:

$$\begin{aligned}\Delta\xi_{21} \sin \alpha_{12} - \Delta\eta_{21} \cos \alpha_{12} &= \\ &\frac{n_{12}}{4g}\left[\left(\Delta W_{\Delta_1} + \Delta W_{\Delta_2}\right)\sin 2\alpha_{12} + \left(\Delta W_{xy_1} + \Delta W_{xy_2}\right)\cos 2\alpha_{12}\right]\end{aligned} \tag{31}$$

This is a very important relationship between gradients from torsion balance measurements, and deflections of the vertical.
Being given a third point P_3 forming a triangle with P_1 and P_2, leads to further two relationships

$$T_{23} = \Delta\xi_{32} \sin \alpha_{23} - \Delta\eta_{32} \cos \alpha_{23} \tag{32}$$

and

$$T_{13} = \Delta\xi_{31} \sin \alpha_{13} - \Delta\eta_{31} \cos \alpha_{13} \tag{33}$$

as in (26).

Proceeding along the triangle formed by P_1 P_2 and P_3 variation of components of deflection of the vertical must be zero, permitting to write further two relationships in addition to the already deduced ones (26), (32) and (33) ; that is:

$$\Delta\xi_{21} + \Delta\xi_{32} + \Delta\xi_{13} = 0 \tag{34}$$

and

$$\Delta\eta_{21} + \Delta\eta_{32} + \Delta\eta_{13} = 0 \tag{35}$$

Thus, for any single triangle, there are six unknowns: $\Delta\xi_{21}$, $\Delta\xi_{32}$, $\Delta\xi_{13}$, $\Delta\xi_{21}$, $\Delta\xi_{32}$, $\Delta\xi_{13}$; for them five, mutually independent equations: (26), (32), (33), (34), (35). may be written. Unambiguous solution to the problem requires further information.

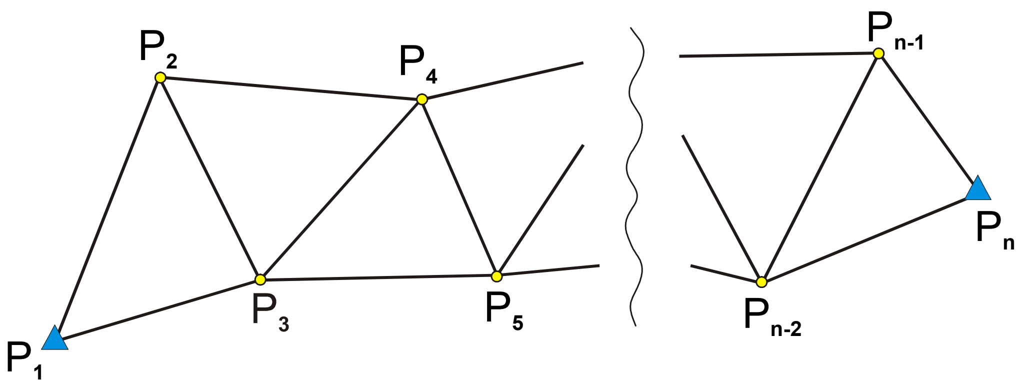

Fig. 3

Now have a look at the interpolation chain of *n* points in *Fig.3*. The *n* points form a chain of *n-2* triangles with *2n-3* triangle sides, each having two unknown components of deflection of the vertical along sides - hence, for all of the network, there is a total of *4n-6* unknowns. While for the *n-2* triangles, *2n-3* equations of the (26) type, and *2n-4* ones of the (34) and (35) types may be written, hence for the *4n-6* unknowns there are *4n-7* equations in all. For an unambiguous solution to the problem, a further information (equation) - independent of those above - is required.

For instance, in case of a chain of interpolation seen in *Fig.3* , if values of components ξ_1 , ξ_n or η_1 , η_n of deflections of the vertical at the two extreme points are known, it may be written

$$\sum_{i=1}^{n-1} \Delta\xi_{i+1,i} = \xi_n - \xi_1 \qquad (36$$

or

$$\sum_{i=1}^{n-1} \Delta\eta_{i+1,i} = \eta_n - \eta_1 \qquad (37$$

So that a total of *4n-6* equations may be written for the *4n-6* unknowns, permitting unambiguous determination of all unknown differences $\Delta\xi$ and $\Delta\eta$ between components of deflection of the vertical.

2. Practical Solution of Interpolation

In those above, fundamentals of interpolation of deflection of the vertical applying torsion balance measurements were considered. Interpolation can be solved by means of various practical computation methods. Every practical solution relies on the fundamentals presented above, but the different computation methods are not equivalent - mainly as to reliability of their respective results. Let us have a look at the practically possible solutions.

Practical solutions belong to two groups. In one variations, $\Delta\xi$ $\Delta\eta$ of components of deflection of the vertical are taken as unknowns, while the other group, components ξ, η of the deflection of the vertical at the points are the required unknowns. In the first case - when differences between the components of the deflection of the vertical between points are taken as unknowns - there are three possibilities of solution:

- inverting the complete coefficient matrix assembled of coefficients of the *4n-6* equations produced by applying (26), (34), (35), (36), (37) type equations, that is, determining *4n-6* unknown values of differences $\Delta\xi$ and $\Delta\eta$ of deflection of the vertical,
- taking the group of the coefficient matrix above referring only to the absolutely necessary *2n-2* unknowns into consideration,
- determining unknowns $\Delta\xi$ $\Delta\eta$ step by step (by successive elimination).

2.1. Traditional Solution Method

The solution method considered as traditional is due to Loránd Eötvös (EÖTVÖS 1906, 1909; SELÉNYI 1953). In this method, in the interpolation nets, the differences of deflections of the vertical between neighboring points are considered as unknowns, writing for the unknowns $\Delta\xi$ and $\Delta\eta$ equations of types (26), (34), (35); as well as (36), or (37). Now, for arbitrary interpolation net (or chain) of *n* points, *4n-6* unknown values of differences $\Delta\xi$ and $\Delta\eta$ of the deflections of the vertical are to be determined. In the preceding item, it was shown that for an unambiguous determination of unknown values $\Delta\xi$ and $\Delta\eta$, the same components of deflections of the vertical hence either ξ or η values at two arbitrary points of the interpolation net (possibly, at end points) are needed. Since in most of the cases, it is not sufficient to know differences $\Delta\xi$, $\Delta\eta$ between neighboring points, but the very ξ, η values at every point are needed, it is insufficient to know one component of the deflection of the vertical at two points of the net, but also the value of the other component at some point should be known. In other words, if the very ξ, η values at points of the interpolation net are to be determined, then, in addition to torsion balance measurements, two known (astrogeodetic) points are needed, with the knowledge of both ξ and η values in one of them, and either the ξ or the η value in the other. Practically, both ξ and η values in the two known astrogeodetic points are available, thus, there is an excess of data, the problem is redundant. In this case, the most probable value of the unknowns is determined by adjustment.

In practice, solution to the adjustment problem is made by using the least squares method.

2.2. Reducing the Number of Unknowns $\Delta\xi$, $\Delta\eta$

Computing interpolation chains by the method in item 2.1 involves much of needless excess work, a drawback both for accuracy and economy of the method. In case of the conventional computation method, excess work consists in inverting, for a chain of *n* points, all coefficient matrices belonging to the *4n-6* unknowns, - although for an unambiguous solution to the problem only *2n-2* unknowns are needed. For a high *n* value, this may significantly reduce accuracy of the interpolated $\Delta\xi$, $\Delta\eta$ values.

To reduce the number of unknowns, let us compose the system of 4n-6 unknowns into two groups. One of the groups contains only the necessary unknowns (for instance, for the chain in *Fig.3*, only the $\Delta\xi$, $\Delta\eta$ values for sides P_1P_2 , P_2P_3 , P_3P_4 , P_4P_5 , ... the other group will contain the needless unknowns (e.g. $\Delta\xi$, $\Delta\eta$ for the remaining sides P_1P_3 , P_2P_4 , P_3P_5 , ...). The other group of unknowns is omitted in the following, and only coefficient matrix of the system constructed of equations for the needed unknowns is to be inverted. This latter is merely of size *(2n-2)* $\times$ *(2n-2)*, hence much less than that of size *(4n-6)* $\times$ *(4n-6)* in the conventional case.

Now let us see what necessary equations are sufficient to be written.
Let us consider *Fig.3* again! Equations (26), (32), (33) yield for the first triangle ($P_1P_2P_3$), eliminating unknowns $\Delta\xi_{31}$ and $\Delta\eta_{31}$:

$$\Delta\xi_{21}\sin\alpha_{12} - \Delta\eta_{21}\cos\alpha_{12} = T_{12} \tag{38}$$

$$\Delta\xi_{32}\sin\alpha_{23} - \Delta\eta_{32}\cos\alpha_{23} = T_{23} \tag{39}$$

$$\Delta\xi_{21}\sin\alpha_{31} - \Delta\eta_{21}\cos\alpha_{31} - \Delta\xi_{32}\sin\alpha_{31} + \Delta\eta_{32}\cos\alpha_{31} = T_{31} \tag{40}$$

while for each of the other triangles further two equations result:

$$\Delta\xi_{i+2,i+1}\sin\alpha_{i+1,i+2} - \Delta\eta_{i+2,i+1}\cos\alpha_{i+1,i+2} = T_{i+1,i+2} \tag{41}$$

and

$$\begin{aligned}\Delta\xi_{i+1,i}\sin\alpha_{i+2,i} - \Delta\eta_{i+1,i}\cos\alpha_{i+2,i} - \\ \Delta\xi_{i+2,i+1}\sin\alpha_{i+2,i} + \Delta\eta_{i+2,i+1}\cos\alpha_{i+2,i} = T_{i+2,i}\end{aligned} \tag{42}$$

where $i = 2, 3, 4, \ldots, n-2$.

These make up *2n-3* equations with *2n-2* unknowns $\Delta\xi$ and $\Delta\eta$. For an unambiguous solution to the problem, in conformity with our previous statements, further information (equation) is needed to obtain from (known) deflections of the vertical at points of the interpolation net. Provided ξ_1 , η_1 and ξ_n , η_n values are given at two arbitrary points of the interpolation chain (possibly at end points), then, in addition to (38), (39), (40), as well as (41), and (42), also conditional equations (36), (37) may be written, and the most probable values of unknowns $\Delta\xi$, $\Delta\eta$ may be determined (by adjustment).

2.3. Interpolation by Successive Elimination

Determining unknowns $\Delta\xi$, $\Delta\eta$ by successive elimination rather than by inverting coefficient matrix of the unknowns offers practical advantages.

To present essentials of the step-wise determination, let us consider again the interpolation chain in *Fig.3*. Irrelevant unknowns (components $\Delta\xi$, $\Delta\eta$ of deflection of the vertical for sides P_1P_3 , P_2P_4 , P_3P_5 , P_4P_6 , ...) will be omitted, only those for sides P_1P_2 , P_2P_3 , P_3P_4 , P_4P_5 , ... are to be determined.

Let us determine first the unknowns for the first side P_1P_2 of triangle $P_1P_2P_3$, starting from the trivial relationship:

$$\Delta\xi_{21} = u = a_1 u + b_1 \tag{43}$$

where

$$a_1 = 1 \quad \text{and} \quad b_1 = 0 \tag{44}$$

By writing equation (43) into (26), and expressing the $\Delta\eta_{12}$ value:

$$\Delta\eta_{12} = \frac{a_1 \sin\alpha_{12}}{\cos\alpha_{12}} u + \frac{b_1 \sin\alpha_{12} - T_{12}}{\cos\alpha_{12}}$$

or concisely:

$$\Delta\eta_{21} = u = c_1 u + d_1 \tag{45}$$

where

$$c_1 = \frac{a_1 \sin\alpha_{12}}{\cos\alpha_{12}}$$

and

$$d_1 = \frac{b_1 \sin\alpha_{12} - T_{12}}{\cos\alpha_{12}} \tag{46}$$

Let us determine further unknowns for the next P_2P_3 side of triangle $P_1P_2P_3$. By eliminating unknowns $\Delta\xi_{31}$ and $\Delta\eta_{31}$ from (26), (32), (33), (34) and (35) for triangle $P_1P_2P_3$ and introducing notation:

$$Q = \left(\sin\alpha_{32}\cos\alpha_{31} - \sin\alpha_{31}\cos\alpha_{23}\right)^{-1} \tag{47}$$

yields for unknowns $\Delta\xi_{32}$ and $\Delta\eta_{32}$:

$$\Delta\xi_{32} = \left(T_{23}\cos\alpha_{31} + T_{31}\cos\alpha_{23} + \Delta\xi_{21}\sin\alpha_{31}\cos\alpha_{23} - \Delta\eta_{21}\cos\alpha_{31}\cos\alpha_{23}\right)Q$$

and

$$\Delta\eta_{32} = \left(T_{23}\sin\alpha_{31} + T_{31}\sin\alpha_{23} + \Delta\xi_{21}\sin\alpha_{31}\sin\alpha_{23} - \Delta\eta_{21}\cos\alpha_{31}\sin\alpha_{23}\right)Q$$

Substituting (43) and (45) :

$$\Delta\xi_{32} = \left[\left(a_1 \sin\alpha_{31} \cos\alpha_{23} - c_1 \cos\alpha_{31} \cos\alpha_{23}\right)u + T_{23} \sin\alpha_{31} + T_{31} \sin_{23} + b_1 \sin\alpha_{31} \cos\alpha_{23} - d_1 \cos\alpha_{31} \cos\alpha_{23}\right]Q$$

and

$$\Delta\eta_{32} = \left[\left(a_1 \sin\alpha_{31} \cos\alpha_{23} - c_1 \cos\alpha_{31} \cos\alpha_{23}\right)u + T_{23} \cos\alpha_{31} + T_{31} \cos_{23} + b_1 \sin\alpha_{31} \sin\alpha_{23} - d_1 \cos\alpha_{31} \sin\alpha_{23}\right]Q$$

or, with other notations:

$$\Delta\xi_{32} = a_2 u + b_2 \tag{48}$$

$$\Delta\eta_{32} = c_2 u + d_2 \tag{49}$$

where

$$a_2 = \left(a_1 \sin\alpha_{31} \cos\alpha_{23} - c_1 \cos\alpha_{31} \cos\alpha_{23}\right)Q$$

$$b_2 = \left(b_1 \sin\alpha_{31} \cos\alpha_{23} - d_1 \cos\alpha_{31} \cos\alpha_{23} + T_{23} \cos\alpha_{31} + T_{31} \cos\alpha_{23}\right)Q \tag{50}$$

$$c_2 = \left(a_1 \sin\alpha_{31} \sin\alpha_{23} - c_1 \cos\alpha_{31} \sin\alpha_{23}\right)Q$$

$$d_2 = \left(b_1 \sin\alpha_{31} \sin\alpha_{23} - d_1 \cos\alpha_{31} \sin\alpha_{23} + T_{23} \sin\alpha_{31} + T_{31} \sin\alpha_{23}\right)Q \tag{51}$$

Coefficients a_i and c_i seem to depend exclusively on the net geometry, while coefficients b_i and d_i on the net geometry and on the second potential derivatives depending on the gradient of the level surface.

Eqs (43), (45), (48) and (49) may be written in turn for all triangles of the chain in *Fig.3*. In general, for the i-th triangle:

$$\Delta\xi_{i+2,i+1} = a_{i+1} u + b_{i+1} \tag{52}$$

$$\Delta\eta_{i+2,i+1} = c_{i+1} u + d_{i+1} \tag{53}$$

leading to a single-parameter system of equations where all unknowns are functions of parameter u.

Like before, to determine parameter u, also here further information is required. Provided that the values of components ξ and η of deflection of the vertical at two extreme points of the net are known, it may be written:

$$\Delta\xi_{n1} = \sum_{i=1}^{n-1} a_i u + \sum_{i=1}^{n-1} b_i \tag{54}$$

and

$$\Delta\eta_{n1} = \sum_{i=1}^{n-1} c_i u + \sum_{i=1}^{n-1} d_i \quad . \tag{55}$$

Value of parameter u may be determined from either (54) or (55). Substituting this u value into (52) and (53) permits to easily determine unknown $\Delta\xi$, $\Delta\eta$ values of differences of deflection of the vertical between all necessary pairs of points.

Simultaneously, by writing (54) and (55), the most probable u value will be obtained by adjustment. To this aim, e.g. the Badekas & Mueller adjustment model suits due to its simplicity (BADEKAS and MUELLER 1967).

In conformity with the principle of this adjustment model, l_i and x_i values with

$$f(l_i, x_i) = 0$$

are to be found, where l_i and x_i are the adjusted values of observed magnitudes, and of the required parameters, respectively. By expanding function f and keeping only first-order terms,

$$f(l_{0i}, x_{0i}) + \frac{\partial f}{\partial l_i} v_i + \frac{\partial f}{\partial x_i} \delta x_i = 0$$

where v_i are the corrections of observed magnitudes l_{0i}, while δx_i the variations of preliminary values x_{0i}; that is:

$$l_i = l_{0i} + v_i$$
$$x_i = x_{0i} + \delta x_i$$

In matrix form:

$$\mathbf{F} + \mathbf{Lv} + \mathbf{Ax} = \mathbf{0}$$

where:

$$\mathbf{F} = \left[f(l_{0i}, x_{0i}) \right] \quad , \quad \mathbf{L} = \left[\frac{\partial f}{\partial l_i} \right] \quad \text{and} \quad \mathbf{A} = \left[\frac{\partial f}{\partial x_i} \right]$$

With this model applied to the problem - that is, to (54) and (55):

$$\mathbf{A} = \begin{bmatrix} \sum_{i=1}^{n-1} a_i \\ \sum_{i=1}^{n-1} c_i \end{bmatrix} \quad , \quad \mathbf{L} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} \sum_{i=1}^{n-1} b_i - \xi_{n1} \\ \sum_{i=1}^{n-1} d_i - \eta_{n1} \end{bmatrix}$$

zeroing the preliminary x_{0i} value (here $x_{0i} = u$). By denoting variances of Σb and Σd by $\mu^2_{\Sigma b}$ and $\mu^2_{\Sigma d}$, weight matrix $\mathbf{P}$, and its inverted $\mathbf{P}^{-1}$ become:

$$\mathbf{P} = \begin{bmatrix} \dfrac{1}{\mu_{\Sigma b}^2} & 0 \\ 0 & \dfrac{1}{\mu_{\Sigma d}^2} \end{bmatrix} \quad , \qquad \mathbf{P}^{-1} = \begin{bmatrix} \mu_{\Sigma b}^2 & 0 \\ 0 & \mu_{\Sigma d}^2 \end{bmatrix} .$$

Let us form now matrix product $\mathbf{S} = \mathbf{L}^* \mathbf{P}^{-1} \mathbf{L}$ ($\mathbf{L}^*$ being transposed of $\mathbf{L}$), then its inverted $\mathbf{S}^{-1}$:

$$\mathbf{S} = \begin{bmatrix} \mu_{\Sigma b}^2 & 0 \\ 0 & \mu_{\Sigma d}^2 \end{bmatrix} \quad , \qquad \mathbf{S}^{-1} = \begin{bmatrix} \dfrac{1}{\mu_{\Sigma b}^2} & 0 \\ 0 & \dfrac{1}{\mu_{\Sigma d}^2} \end{bmatrix} .$$

With the above notations, the solution in general form is:

$$x = -(\mathbf{A}^* \mathbf{S}^{-1} \mathbf{A})^{-1} \mathbf{A}^* \mathbf{S}^{-1} \mathbf{F} \quad ,$$

in the actual case:

$$u = \frac{\sum_{i=1}^{n-1} a_i \left(\sum_{i=1}^{n-1} b_i - \xi_{n1} \right) \mu_{\Sigma d}^2 + \sum_{i=1}^{n-1} c_i \left(\sum_{i=1}^{n-1} d_i - \eta_{n1} \right) \mu_{\Sigma b}^2}{\left(\sum_{i=1}^{n-1} a_i \right)^2 \mu_{\Sigma d}^2 + \left(\sum_{i=1}^{n-1} c_i \right)^2 \mu_{\Sigma b}^2}$$

or by denoting solutions of (54) and (55) by u_ξ and u_η , respectively:

$$u = \frac{\left(\sum_{i=1}^{n-1} a_i \right)^2 \mu_{\Sigma d}^2 u_\xi + \left(\sum_{i=1}^{n-1} c_i \right)^2 \mu_{\Sigma b}^2 u_\eta}{\left(\sum_{i=1}^{n-1} a_i \right)^2 \mu_{\Sigma d}^2 + \left(\sum_{i=1}^{n-1} c_i \right)^2 \mu_{\Sigma b}^2} \tag{56}$$

Resubstituting this u value into (54) and (55), ξ and η values computed with values of Σa , Σb , Σc , Σd will generally deviate from the difference of components of deflection of the vertical given between extreme points. The resulting misclosures are considered with opposite signs as corrections and distributed between terms of the given sums Σb and Σd, according to their variances,- where covariances of terms b_i and d_i are assumed to be negligible.

2.4. Direct Computation of Components ξ , η

The practical solutions above are more or less advantageous to be applied to interpolation chains (e.g. that in *Fig.3*) with known values of deflection of the vertical at the beginning and end points. Application of the same solution methods may involve unpredictable computational difficulties if interpolation is not made along a chain but for points of an arbitrary, extensive triangulation network. Although writing intermediary equations of the (26), (32), (33) type represents no problem, but it is rather intricate to generate constraining condition equations (34), (35) by a computer. If moreover, the net includes more than two astrogeodetic points with given ξ, η values, then computer generation of the constraining condition equations is rather problematic; during processing, the computer program may get into an infinite cycle. To clear and solve similar problems, graph theory considerations are needed (TAKÁTSY, 1985).

All these difficulties may be overcome by considering ξ, η values of deflection of the vertical at the point as unknowns in interpolating rather than differences $\Delta\xi, \Delta\eta$ between the points. Accordingly, let us transform (26) -type relationships by substituting:

$$\Delta\xi_{ij} = \xi_i - \xi_j$$
$$\Delta\eta_{ij} = \eta_i - \eta_j$$

to

$$T_{ij} = \xi_j \sin\alpha_{ij} + \eta_j \cos\alpha_{ij} - \xi_i \sin\alpha_{ij} - \eta_i \cos\alpha_{ij} \quad . \tag{57}$$

This significantly reduces the number of unknowns, namely, there will be two unknowns for each point rather than per side. (In an arbitrary network, there are much less of points than of sides, since according to the classic principle of triangulation, every new point joins the existing network by two sides. For a homogeneous triangulation network, the side/point ratio may be higher than two.) Another of its advantages is that there is no requirement for writing constraining conditions (34), (35) for the triangles, they being contained in the established observation equations. For an interpolation net with m astrogeodetic points with known values of deflection of the vertical, with the relevant constraints the number of unknowns may be further reduced, with an additional size reduction of the normal equations matrix.

Let us see now, how to solve interpolation for an arbitrary network with more of astrogeodetic points than needed for an unambiguous solution, where components of deflection of the vertical are known, and the ξ, η values are determined by adjustment. Relation between torsion balance measurements W_Δ and W_{xy} and unknown ξ, η values of the deflection of the vertical is obtained from (57):

$$T_{ij} = \frac{n_{ij}}{4g}\left[\left((W_\Delta - U_\Delta)_i + (W_\Delta - U_\Delta)_j\right)\sin 2\alpha_{ij} + \left((W_{xy} - U_{xy})_i + (W_{xy} - U_{xy})_j\right)\cos 2\alpha_{ij}\right] \tag{58}$$

where U_Δ and U_{xy} being normal values of gradients. The question arises what data are to be considered as measurement results for adjustment: the real torsion balance measurements W_Δ and W_{xy} , or T_{ij} values from (58). Since no simple functional relationship (observation equation) with a measurement result on one side, and unknowns on the other side of an equation can be written, computation ought to be made under conditions of adjustment of direct measurements, rather than with measured unknowns (according to adjustment group *V*) - this is, however, excessively demanding for computation, requiring excessive storage capacity. Hence concerning measurements, two approximations will be applied: on the one hand, components of deflection of the vertical measured at astrogeodetic points are left uncorrected - thus, they are input to adjustment as constraints,- on the other hand, magnitudes T_{ij} on the left hand side of fundamental equation (57) are considered as fictitious measurements and corrected. Thereby observation equation (57) becomes:

$$T_{ij} + v_{ij} = \xi_j \sin\alpha_{ij} + \eta_j \cos\alpha_{ij} - \xi_i \sin\alpha_{ij} - \eta_i \cos\alpha_{ij} \tag{59}$$

permitting computation under conditions given by adjusting indirect measurements between unknowns (adjustment group *IV*).

The first approximation is possible since reliability of the components of deflection of the vertical determined from astrogeodetic measurements exceeds that of the interpolated values considerably (a principle applied also to geodetic basic networks). Validity of the second approximation will be reconsidered in connection with the problem of weighting.

For every triangle side of the interpolated net, observation equation relying on (59):

$$v_{ij} = \xi_j \sin\alpha_{ij} + \eta_j \cos\alpha_{ij} - \xi_i \sin\alpha_{ij} - \eta_i \cos\alpha_{ij} - T_{ij} \tag{60}$$

may be written. In matrix form:

$$\underset{(m,1)}{\mathbf{v}} = \underset{(m,2n)}{\mathbf{A}}\,\underset{(2n,1)}{\mathbf{x}} + \underset{(m,1)}{\mathbf{l}}$$

where $\mathbf{A}$ is the coefficient matrix of observation equations, $\mathbf{x}$ is the vector containing unknowns ξ and η, $\mathbf{l}$ is the vector of constant terms; m is the number of sides in the interpolation net; and n is the number of points. Non-zero terms in an arbitrary row i of matrix $\mathbf{A}$ are:

$$\begin{bmatrix} \dots & \sin\alpha_{ij} & \cos\alpha_{ij} & -\sin\alpha_{ij} & -\cos\alpha_{ij} & \dots \end{bmatrix} \tag{61}$$

while vector elements of constant term $\mathbf{l}$ are the T_{ij} values.

Constraint values of deflection of the vertical fixed at astrogeodetic points modify the structure of observation equations. Be

$$\xi_k = \xi_{kc} = given, \quad k = 1, 2, \dots, m_1$$
$$\eta_k = \eta_{kc} = given, \quad k = 1, 2, \dots, m_2$$

given values of deflection of the vertical. Substituting them into observation equations (60) reduces the number of unknowns, modifying coefficient matrix $\mathbf{A}$ and constant term vector $\mathbf{l}$ of observation equations. If, for instance, in (59), $\xi_i = \xi_{ic} = given,$ then the corresponding row (61) of matrix $\mathbf{A}$ is:

$$\begin{bmatrix} \dots & \sin\alpha_{ij} & \cos\alpha_{ij} & \dots & -\cos\alpha_{ij} & \dots \end{bmatrix}$$

the changed constant term being: $T_{ij} + \xi_{ic} \sin\alpha_{ij}$; that is columns of ξ_i , and of coefficients of ξ_i are missing from vector **x** , and matrix **A** , respectively, while corresponding terms of constant term vector **l** are changed by a value $\xi_{ic} \sin\alpha_{ij}$. In an interpolation net, at certain points, ξ values, at other points η values may be given. However, at the same astrogeodetic point, both ξ and η values are usually known. In this case, coefficient matrix **A** , vector **x** , and constant term vector **l** of observation equations are further modified , as described above.

Adjustment raises also the problem of weighting. Earlier the approximation comprised - rather than direct torsion balance measurements - starting from fictive measurements produced from them. Fictive measurements may only be applied, however, if certain conditions are met. The most important condition is the deducibility of covariance matrix of fictive measurements from the law of error propagation, requiring, however, a relation yielding fictive measurement results, - in the actual case, *Eq.* (58). Among quantities on the right-hand side of (58), torsion balance measurements W_Δ and W_{xy} may be considered as wrong. They are about equally reliable ($\pm 1E$), furthermore, they may be considered as mutually independent quantities, thus, their weighting coefficient matrix $\mathbf{Q_{WW}}$ will be a unit matrix. With the knowledge of $\mathbf{Q_{WW}}$, weighting coefficient matrix $\mathbf{Q_{TT}}$ of fictive measurements T (after DETREKŐI 1991) is:

$$\mathbf{Q_{TT}} = \mathbf{F}^*\mathbf{Q_{WW}}\mathbf{F} = \mathbf{F}^*\mathbf{F}$$

$\mathbf{Q_{WW}} = \mathbf{E}$ being a unit matrix. Elements of an arbitrary row i of matrix $\mathbf{F}^*$ are:

$$\left[\left(\frac{\partial T_{ij}}{\partial W_\Delta}\right)_1 \left(\frac{\partial T_{ij}}{\partial W_\Delta}\right)_2 \;,\ldots,\; \left(\frac{\partial T_{ij}}{\partial W_\Delta}\right)_n \left(\frac{\partial T_{ij}}{\partial W_{xy}}\right)_1 \left(\frac{\partial T_{ij}}{\partial W_{xy}}\right)_2 \;,\ldots,\; \left(\frac{\partial T_{ij}}{\partial W_{xy}}\right)_n\right]$$

For the following considerations let us produce rows $\mathbf{f}_1^*$ and $\mathbf{f}_2^*$ of matrix $\mathbf{F}^*$ (referring to sides between points $P_1 - P_2$ and $P_1 - P_3$ respectively):

$$\mathbf{f}_1^* = \left[n_{12}k\left(\sin 2\alpha_{12}\ , \sin 2\alpha_{12}\ , 0\ , 0\ , \ldots , 0\ , \cos 2\alpha_{12}\ , \cos 2\alpha_{12}\ , 0\ , 0\ , \ldots , 0\ ,\right)\right]$$

and

$$\mathbf{f}_2^* = \left[n_{13}k\left(\sin 2\alpha_{13}\ , 0\ , \sin 2\alpha_{13}\ , 0\ , 0\ , \ldots , 0\ , \cos 2\alpha_{13}\ , 0\ , \cos 2\alpha_{13}\ , 0\ , 0\ , \ldots , 0\ ,\right)\right]$$

where $k = 1/4g$ is constant. Using $\mathbf{f}_1^*$, variance of T value referring to side $P_1 - P_2$ is:

$$m^2 = n_{12}^2 k^2 \left(2\sin^2 2\alpha_{12} + 2\cos^2 2\alpha_{12}\right) = 2k^2 n_{12}^2$$

while $\mathbf{f}_1^*$ and $\mathbf{f}_2^*$ yield covariance of T values for sides $P_1 - P_2$ and $P_1 - P_3$:

$$\operatorname{cov} = n_{12} n_{13} k^2 \left(\sin 2\alpha_{12} \sin 2\alpha_{13} + \cos 2\alpha_{12} \cos 2\alpha_{13}\right)\ .$$

Thus, fictive measurements may be stated to be correlated, and the weighting coefficient matrix contains covariance elements at the junction point of the two sides. If needed, the weighting matrix may be produced by inverting this weighting coefficient matrix. Practically, however, two approximations are possible: either fictive measurements T are considered to be mutually independent, so weighting matrix is a diagonal matrix; or fictive measurements are weighted in inverted quadratic relation to the distance.

By assuming independent measurements, the second approximation results also from inversion, since terms in the main diagonal of the weighting coefficient matrix are proportional to the square of the side lengths. The neglection is, however, justified, in addition to the simplification of computation, also by the fact that contradictions are due less to measurement errors than to functional errors of the computational model (to be discussed later).

2.5. Interpolation for Corner Points of a Square Net

This interpolation method for an extensive area, developed by János Renner (RENNER 1952, 1956, 1957) also requires inversion of all the coefficient matrix.
The gist of Renner's method is to determine values of deflection of the vertical at corner points of an arbitrary square net rather than at torsion balance measurement points. To this aim, the considered area is covered by a square net with 1 to 2 km side length, of N-S and E-W lines, and the needed values of gradients W_Δ and W_{xy} are interpolated for the resulting corner points relying on known torsion balance measurements.

Any inner point of the square net is surrounded by eight neighboring points as seen in *Fig.4* , forming eight rectangular triangles giving rise to rather simple relationships for components $\Delta\xi$, $\Delta\eta$ of deflection of the vertical at the mid-point.

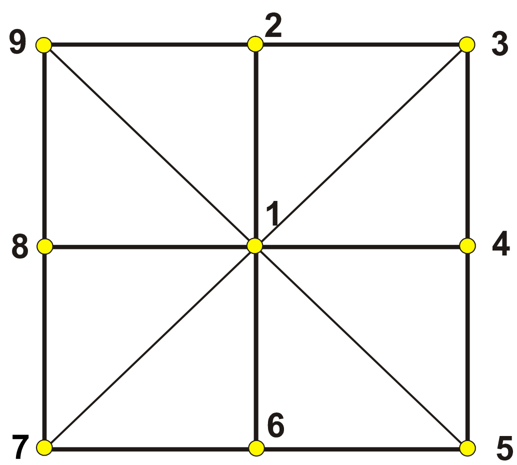

Fig. 4

Writing these equations for every point of the square net, each relationship for differences $\Delta\xi$, $\Delta\eta$ occurs twice, hence, instead of eight equations per point there are four mutually independent equations.
In his test computations, Renner considered the $\Delta\xi$, $\Delta\eta$ values as unknowns, but it is more convenient to take the ξ, η values themselves as unknowns. Now, for eight points $P_2 \div P_9$ surrounding an arbitrary point of the interpolation net (e.g. P_1 in *Fig.4*), the following rather simple equations may be written:

$$
\begin{aligned}
T_{12} &= \eta_2 - \eta_1 \\
\sqrt{2} T_{13} &= \xi_3 + \eta_3 - \xi_1 - \eta_1 \\
T_{14} &= \xi_4 - \xi_1 \\
\sqrt{2} T_{15} &= \xi_5 - \eta_5 - \xi_1 + \eta_1 \\
T_{16} &= -\eta_6 + \eta_1 \\
\sqrt{2} T_{17} &= -\xi_7 - \eta_7 + \xi_1 + \eta_1 \\
T_{18} &= -\xi_8 + \xi_1 \\
\sqrt{2} T_{19} &= -\xi_9 + \eta_9 + \xi_1 - \eta_1
\end{aligned}
$$

Similarly, also T_{ij} values on the left-hand side of the equations are simple to compute, namely, values of trigonometrical functions in T_{ij} cannot be other than 0 or 1. For any interpolation net of arbitrary size, only these eight relationships may be written, except in the surrounding of astrogeodetic points including constraining values ξ, η, due to their junction.

2. 6. Application of the Matrix Orthogonalization Method

In any practical solution other than the method of successive elimination, in applying the conventional adjustment procedure, difficulties in inverting a rather large-size matrix may emerge. There are essentially two ways of adjustment in some problem: either by the usual method of establishing and solving normal equations, or directly, by the matrix orthogonalization method.

Solution of certain adjustment problems by the usual method - establishing and inverting normal equations - fails a result of expected accuracy, because e.g. the coefficient matrix of the arising normal equations is poorly conditioned. So practical solution to adjustment problems is advisably done by the matrix orthogonalization method, avoiding to establish normal equations, and the required, numerically more stable solution is directly obtained by applying proper matrix transformations (VÖLGYESI 1975, 1979, 1980).

The quite simple principle of the matrix orthogonalization adjustment method is illustrated by the hypermatrix transformation,

$$
\left[\begin{array}{c|c} \underset{(n,r)}{\mathbf{A}} & \underset{(n,1)}{\mathbf{l}} \\ \hline \underset{(r,r)}{\mathbf{E}} & \underset{(r,1)}{\mathbf{0}} \end{array}\right] \longrightarrow \left[\begin{array}{c|c} \underset{(n,r)}{\mathbf{W}} & \underset{(n,1)}{\mathbf{v}} \\ \hline \underset{(r,r)}{\mathbf{G}^{-1}} & \underset{(r,1)}{\mathbf{x}} \end{array}\right] \qquad (62)
$$

where $\mathbf{A}$ is the coefficient matrix of observation equations, $\mathbf{l}$ is the vector of constant terms, $\mathbf{E}$ is a unit matrix, $\mathbf{O}$ is a zero vector, $\mathbf{W}$ is a matrix with orthonormal columns, and $\mathbf{G}^{-1}$ is an upper triangular matrix.

To interpret algorithm of transformation (62), let us introduce notations: $\mathbf{a_i}$ is the column i of matrix $\mathbf{A}$; $\mathbf{w_i}$ is the column i of matrix $\mathbf{W}$; $\mathbf{e_i}$ is the column i of matrix $\mathbf{E}$; and $\mathbf{g_i}$ is the column i of matrix $\mathbf{G^{-1}}$. With these notations, matrix transformation (62) comprises the following steps:

$$\begin{bmatrix}\mathbf{w_1}\\\mathbf{g_1}\end{bmatrix}=\frac{\begin{bmatrix}\mathbf{a_1}\\\mathbf{e_1}\end{bmatrix}}{\left\|\mathbf{a_1}\right\|_E}$$

$$\left\{\begin{array}{l}\left.\begin{array}{l}\begin{bmatrix}\mathbf{a_i}\\\mathbf{e_i}\end{bmatrix}_{<k>}=\begin{bmatrix}\mathbf{a_i}\\\mathbf{e_i}\end{bmatrix}\\\begin{bmatrix}\mathbf{a_i}\\\mathbf{e_i}\end{bmatrix}_{<k+1>}=\begin{bmatrix}\mathbf{a_i}\\\mathbf{e_i}\end{bmatrix}_{<k>}-\left((\mathbf{a_i})_{<k>},\mathbf{w_k}\right)\begin{bmatrix}\mathbf{w_k}\\\mathbf{g_k}\end{bmatrix}\\\begin{bmatrix}\mathbf{w_i^*}\\\mathbf{g_i^*}\end{bmatrix}=\begin{bmatrix}\mathbf{a_i}\\\mathbf{e_i}\end{bmatrix}_{<i>}\end{array}\right\}\\\begin{bmatrix}\mathbf{w_i}\\\mathbf{g_i}\end{bmatrix}=\frac{\begin{bmatrix}\mathbf{w_i^*}\\\mathbf{g_i^*}\end{bmatrix}}{\left\|\mathbf{w_i^*}\right\|_E}\end{array}\right.$$

$$i=2,3,...,r;k=1,2,...,j-1$$

then:

$$\begin{bmatrix}\mathbf{v}\\\mathbf{x}\end{bmatrix}=\begin{bmatrix}\mathbf{l}\\\mathbf{0}\end{bmatrix}-\sum_{k=1}^{r}\left(\mathbf{l},\mathbf{w_k}\right)\begin{bmatrix}\mathbf{w_k}\\\mathbf{g_k}\end{bmatrix}$$

where $\left\|\mathbf{a_1}\right\|_E$ and $\left\|\mathbf{w_i^*}\right\|_E$ are Euclidean norms of column, vectors $\mathbf{a_1}$, and $\mathbf{w_i^*}$, respectively, while $\left(\mathbf{a_i},\mathbf{w_k}\right)$ and $\left(\mathbf{l},\mathbf{w_k}\right)$ are scalar products of column vectors $\mathbf{a_i}$ and $\mathbf{w_k}$, and of vectors $\mathbf{l}$ and $\mathbf{w_k}$, respectively.

Matrix transformation (62) directly yields the wanted unknowns x_i and corrections v_i in place of vector $\mathbf{x}$ and $\mathbf{v}$ respectively (VÖLGYESI 1979, 1980).

Variances and covariances of unknowns x_i are comprised in weight coefficient matrix

$$\mathbf{Q_{(x)}}=\mathbf{G^{-1}}(\mathbf{G^{-1}})^{*} \tag{63}$$

where $(\mathbf{G^{-1}})^{*}$ is transposed of $\mathbf{G^{-1}}$.

3.The Reliability of Interpolation

Different practical solution methods of interpolation do not yield equally reliable values of deflection of the vertical. There are several possibilities to describe reliability, to determine mean errors of interpolated values.

The simplest method yielding the most realistic information on reliability is direct comparison of interpolated values to known values of deflection of the vertical. This is feasible if there is a relatively dense net of astrogeodetic points, and some astrogeodetic points within the interpolation net may be handled as unknown (control) points, where interpolated values of deflection of the vertical may be directly compared to astrogeodetic values. There is another, again simple possibility to check reliability of interpolation methods by creating different interpolation nets (chains) joining at common net points. Interpolated values should be more or less equal at identical points of different nets - obviously, the rate of deviations is characteristic of the reliability of interpolation.

If there is no possibility to directly check interpolated values, then reliability of the interpolated values may also be determined by mathematical methods, relying on laws of error propagation.

In applying the conventional adjustment method, mean errors of the interpolated values of deflection of the vertical may be determined by the method known from the variance-covariance matrix

$$\mathbf{M}_{(\mathbf{x})} = \mu_0^2 \mathbf{Q}_{(\mathbf{x})}$$

where μ_0^2 is the standard error of unit weight, while $\mathbf{Q}_{(\mathbf{x})}$ is the weighting coefficient matrix of unknown deflections of the vertical (DETREKŐI 1991). Matrix $\mathbf{Q}_{(\mathbf{x})}$ is either simply the inverse $\mathbf{N}^{-1}$ of the coefficient matrix of normal equations, or, in more complex cases, it is simple to compute by using $\mathbf{N}^{-1}$.

Reliability indices of interpolated values of deflections of the vertical can also be simply obtained by making the computation by the matrix orthogonalization method. In this case, weighting coefficient matrix $\mathbf{Q}_{(\mathbf{x})}$ of interpolated deflections of the vertical may be computed according to (63).

Compared to the case above, a more detailed consideration will be given to reliability indices of results obtained by the successive elimination method. Here, too, our essential problem is to deduce the reliability of interpolated deflections of the vertical from reliability indices of starting data. Our examinations apply the general law of error propagation. Let multivariate functions:

$$\left.\begin{aligned} u &= f(x, y, z, \ldots) \\ v &= g(x, y, z, \ldots) \\ w &= h(x, y, z, \ldots) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\}$$

be given, just as:

$$M = \begin{bmatrix} \mu_x^2 & c_{xy} & c_{xz} & \ldots \\ c_{yx} & \mu_y^2 & c_{yz} & \ldots \\ c_{zx} & c_{zy} & \mu_{xz}^2 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{bmatrix}$$

where μ_i^2 is the variance (mean square error) of variable i, and c_{ij} is the covariance of independent variables i and j:

$$c_{ij} = r_{ij} |\mu_i| |\mu_j|$$

r_{ij} is the correlation coefficient between variables i and j. Applying notation

$$\mathbf{F}^* = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} & \cdots \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} & \cdots \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

($\mathbf{F}^*$ is transposed of $\mathbf{F}$), the required variance-covariance matrix

$$\mathbf{N} = \begin{bmatrix} \mu_u^2 & c_{uv} & c_{uw} & \cdots \\ c_{vu} & \mu_v^2 & c_{vw} & \cdots \\ c_{wu} & c_{wv} & \mu_w^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

of magnitudes $u, v, w, \ldots$ is:

$$\mathbf{N} = \mathbf{F}^* \mathbf{M} \mathbf{F} \quad . \tag{64}$$

Let us consider values $\mu^2_{W_\Delta}$, $\mu^2_{W_{xy}}$ and c_{W_Δ}, $c_{W_{xy}}$, for torsion balance measurements, and $\mu^2_{\xi_0}$ and $\mu^2_{\eta_0}$ for known deflections of the vertical at astrogeodetic points as being given. Errors of distances and azimuths in (30) and (31) computed from co-ordinates of measurement points being negligible compared to errors of torsion balance measurements (VÖLGYESI 1975, 1976), hence applying those above to the sense:

$$\mu^2_{T_{12}} = \left(\frac{n_{12}}{2g}\right)^2 \left[2\sin^2 2\alpha_{12}\ \mu^2_{W_\Delta} + 2\cos^2 2\alpha_{12}\ \mu^2_{W_{xy}} + 4\sin 2\alpha_{12} \cos 2\alpha_{12}\ c_{W_\Delta, W_{xy}}\right]$$

$$\mu^2_{T_{23}} = \left(\frac{n_{23}}{2g}\right)^2 \left[2\sin^2 2\alpha_{23}\ \mu^2_{W_\Delta} + 2\cos^2 2\alpha_{23}\ \mu^2_{W_{xy}} + 4\sin 2\alpha_{23} \cos 2\alpha_{23}\ c_{W_\Delta, W_{xy}}\right]$$

$$\vdots$$

$$c_{T_{23}, T_{31}} = \frac{n_{23} n_{31}}{(2g)^2} [\ \sin 2\alpha_{23} \sin 2\alpha_{31}\ \mu^2_{W_\Delta} + \cos 2\alpha_{23} \cos 2\alpha_{31}\ \mu^2_{W_{xy}} + \sin(2\alpha_{23} + 2\alpha_{31})\ c_{W_\Delta, W_{xy}}\ ]$$

$$\vdots$$

From those above, according to (44), (46), (50) and (51), applying notations in (47):

$$\mu_{b_1}^2 = 0$$

$$\mu_{d_1}^2 = \frac{\mu_{T_{12}}^2}{\cos^2 \alpha_{12}}$$

$$c_{b_1,d_1} = 0$$

$$\begin{aligned}\mu_{b_2}^2 = [\ & \cos^2 \alpha_{23}\ \mu_{T_{31}}^2 + \cos^2 \alpha_{31}\ \mu_{T_{23}}^2 + 2\cos\alpha_{23} \cos\alpha_{31}\ c_{T_{23},T_{31}} + \\ & \sin^2 \alpha_{31} \cos^2 \alpha_{23}\ \mu_{b_1}^2 + \cos^2 \alpha_{31} \cos^2 \alpha_{23}\ \mu_{d_1}^2 - \\ & 2\sin\alpha_{31} \cos\alpha_{31} \cos^2 \alpha_{23}\ c_{b_1,d_1}\ ] Q^2\end{aligned}$$

$$\begin{aligned}\mu_{d_2}^2 = [\ & \sin^2 \alpha_{23}\ \mu_{T_{31}}^2 + \sin^2 \alpha_{31}\ \mu_{T_{23}}^2 + 2\sin\alpha_{23} \sin\alpha_{31}\ c_{T_{23},T_{31}} + \\ & \sin^2 \alpha_{31} \sin^2 \alpha_{23}\ \mu_{b_1}^2 + \cos^2 \alpha_{31} \sin^2 \alpha_{23}\ \mu_{d_1}^2 - \\ & 2\sin\alpha_{31} \cos\alpha_{31} \sin^2 \alpha_{23}\ c_{b_1,d_1}\ ] Q^2\end{aligned}$$

$$\begin{aligned}c_{b_2,d_2} = [\ & \sin\alpha_{23} \cos\alpha_{23}\ \mu_{T_{31}}^2 + \sin\alpha_{31} \cos\alpha_{31}\ \mu_{T_{23}}^2 + \\ & (\cos\alpha_{32} \sin\alpha_{31} + \sin\alpha_{31} \cos\alpha_{31})\ c_{T_{23},T_{31}} + \\ & \sin^2 \alpha_{31} \sin\alpha_{23} \cos\alpha_{23}\ \mu_{b_1}^2 + \cos^2 \alpha_{31} \sin\alpha_{23} \cos\alpha_{23}\ \mu_{d_1}^2 - \\ & 2\sin\alpha_{31} \cos\alpha_{31} \sin\alpha_{23} \cos\alpha_{23}\ c_{b_1,d_1}\ ] Q^2\end{aligned}$$

$$\vdots$$

ultimately yielding:

$$\mu_{\Sigma b}^2 = \mu_{b_1}^2 + \mu_{b_2}^2 + \ldots + \mu_{b_{n-1}}^2 + c_{b_1,b_2} + \ldots$$
$$\mu_{\Sigma d}^2 = \mu_{d_1}^2 + \mu_{d_2}^2 + \ldots + \mu_{d_{n-1}}^2 + c_{d_1,d_2} + \ldots$$

Thereby one main goal to obtain variances $\mu_{\Sigma b}^2$, and $\mu_{\Sigma d}^2$ needed for (56) has been achieved.

At last, let us determine mean errors of values of deflections of the vertical obtained by successive interpolation. Variance of parameter u from (54) or (55) is:

$$\mu_u^2 = \frac{\mu_{\xi 0}^2 + \mu_{\Sigma b}^2}{\left(\sum_{i=1}^{n-1} a_i \right)^2}$$

or

$$\mu_u^2 = \frac{\mu_{\eta 0}^2 + \mu_{\Sigma d}^2}{\left(\sum_{i=1}^{n-1} c_i \right)^2}$$

depending on what data are known for determining u . According to (52) and (53), using hitherto results:

$$\mu^2_{\Delta\xi_{i+1,i}} = a^2_{i+1}\mu^2_u + \mu^2_{b_{i+1}}$$
$$\mu^2_{\Delta\eta_{i+1,i}} = c^2_{i+1}\mu^2_u + \mu^2_{d_{i+1}}$$

are variances of the differences of deflections of the vertical. In final account, mean errors of the required components of the deflection of the vertical are:

$$\mu_{\xi_i} = \pm\left[\mu^2_{\xi 0} + \left(\sum_{k=1}^{i} a_k\right)^2 \mu^2_u + \mu^2_{\Sigma b}\right]^{\frac{1}{2}} \tag{65}$$

$$\mu_{\eta_i} = \pm\left[\mu^2_{\eta 0} + \left(\sum_{k=1}^{i} c_k\right)^2 \mu^2_u + \mu^2_{\Sigma d}\right]^{\frac{1}{2}} \tag{66}$$

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Dr. Lajos VÖLGYESI, Department of Geodesy and Surveying, Budapest University of Technology and Economics, H-1521 Budapest, Hungary, Műegyetem rkp. 3.
Web: http://sci.fgt.bme.hu/volgyesi E-mail: volgyesi@eik.bme.hu