# COMPARISON OF INTERPOLATION AND COLLOCATION TECHNIQUES USING TORSION BALANCE DATA

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## Abstract

**First the relevant theory of the interpolation and collocation methods, both used here for the recovery of deflections of the vertical and geoid heights from torsion balance data is discussed. We have selected a mostly flat area in Hungary where all kind of torsion balance measurements are available at 249 points. There were 3 astrogeodetic points providing initial data for the interpolation, and there were geoid heights at 10 checkpoints interpolated from an independent gravimetric geoid solution. The size of our test area is about 800 km$^2$ and the average site distance of torsion balance data is 1.5 - 2 km. The interpolation method provided a least squares solution for deflections of the vertical and geoid heights at all points of the test network. By collocation two independent solutions were computed from $W_{zx}$ , $W_{zy}$ and $W_{yy} - W_{xx}$ , $2W_{xy}$ gradients for all the above, using astrogeodetic data to achieve a complete agreement at these sites with the interpolation method. These two solutions agreed at the cm level for geoid heights. The standard deviation of geoid height differences at checkpoints were about ±1-3 cm. The $W_{yy} - W_{xx}$ , $2W_{xy}$ combination (i.e. pure horizontal gradients) yielded better results since the maximum geoid height difference was only 3.6 cm. The differences in the deflection components were generally below 1”, slightly better for the $\eta$ component. The above results confirm the fact that torsion balance measurements give good possibility to compute very precise local geoid heights at least for flat areas.**

## 1. INTERPOLATION OF DEFLECTION OF THE VERTICAL

**A very simple relationship based on potential theory can be written for the changes of $\Delta\xi_{ik}$ and $\Delta\eta_{ik}$ between arbitrary points $i$ and $k$ of the deflection of the vertical components $\xi$ and $\eta$ as well as for gravity gradients $W_\Delta = W_{yy} - W_{xx}$ and $2W_{xy}$ measured by torsion balance:**

$$\Delta\xi_{ik} \sin\alpha_{ki} - \Delta\eta_{ik} \cos\alpha_{ki} =$$

$$\frac{s_{ik}}{4g} \left\{ \left[ \left(W_\Delta - U_\Delta\right)_i + \left(W_\Delta - U_\Delta\right)_k \right] \sin 2\alpha_{ik} + \left[ \left(W_{xy} - U_{xy}\right)_i + \left(W_{xy} - U_{xy}\right)_k \right] 2\cos 2\alpha_{ik} \right\} \quad (1)$$

where $W_\Delta = W_{yy} - W_{xx}$ , $U_\Delta = U_{yy} - U_{xx}$ , $s_{ik}$ is the distance between points $i$ and $k$ , $g$ is the average value of gravity between them, $U_{xx}$ , $U_{yy}$ and $U_{xy}$ are gravity gradients in the normal gravity field, whereas $\alpha_{ik}$ is the azimuth between the two points (Völgyesi 1993, 1995). Writing the left side of Eq. (1) in other form it follows:

$$\xi_k \sin\alpha_{ki} - \xi_i \sin\alpha_{ki} - \eta_k \cos\alpha_{ki} + \eta_i \cos\alpha_{ki} =$$
$$\frac{s_{ik}}{4g}\left\{\left[\left(W_\Delta - U_\Delta\right)_i + \left(W_\Delta - U_\Delta\right)_k\right]\sin 2\alpha_{ik} + \left[\left(W_{xy} - U_{xy}\right)_i + \left(W_{xy} - U_{xy}\right)_k\right]2\cos 2\alpha_{ik}\right\} \tag{2}$$

The computation being fundamentally an integration, practically possible only by approximation, in deriving (1) or (2) it had to be assumed that the change of gravity gradients between points $i$ and $k$ , measured by torsion balance, was linear – thus the equality sign in (1) or (2) is valid only for this case (Völgyesi 1993).

## 2. COMPUTATION OF LOCAL GEOID HEIGHTS

Geoid undulation difference $\Delta N_{ik}$ can be computed from $\xi$ , $\eta$ components interpolated by (2) between points $P_i$ and $P_k$ using the method of astronomical levelling:

$$\Delta N_{ik} = \left(\frac{\xi_i + \xi_k}{2}\cos\alpha_{ik} + \frac{\eta_i + \eta_k}{2}\sin\alpha_{ik}\right)s_{ik} \quad . \tag{3}$$

To eliminate an important problem of classical computation of astronomical levelling we used the original torsion balance measurement points directly for the geoid computation instead of regular grid points - as it was suggested earlier (Völgyesi, 1998, 2001). In this case we use a net of triangles instead of squares, and (3) gives the relationship between components of deflection of the vertical $\xi$ , $\eta$ and the geoid height change $\Delta N$ for each triangle sides in an arbitrary azimuth $\alpha$ . To reduce the number of unknowns we considered geoid heights $N$ directly as unknowns instead of differences $\Delta N$ for a pair of network points:

$$N_k - N_i = \left(\frac{\xi_i + \xi_k}{2}\cos\alpha_{ik} + \frac{\eta_i + \eta_k}{2}\sin\alpha_{ik}\right)s_{ik} \quad . \tag{4}$$

This significantly reduces the number of unknowns, namely, there will be one unknown for each point rather than per triangle side. In an arbitrary network, *there are much less of points than of sides*, since according to the classic principle of triangulation, every new point joins the existing network by two sides. For a homogeneous triangulation network, the *side/point* ratio may be higher than two. Moreover in this case writing constraints (going around each triangles of network the sum of $\Delta N$ differences for the three sides must be zero) is not required for the triangles, they being contained in the established observation equations (4). For an interpolation net with $m$ points with known geoid heights, with the relevant constraints the number of unknowns may be further reduced, with an additional size reduction of the matrix of normal equations.

**Let us see now, how to complete computation for an arbitrary network with more points than needed for an unambiguous solution, where initial geoid heights are known. In this case the unknown $N$ values are determined by adjustment. A relation between components of deflection of the vertical $\xi$, $\eta$ and unknown geoid heights $N$ can be obtained from (4), where**

$$C_{ik} = \left( \frac{\xi_i + \xi_k}{2} \cos\alpha_{ik} + \frac{\eta_i + \eta_k}{2} \sin\alpha_{ik} \right) s_{ik} \tag{5}$$

**is constant for each triangle side. The question arises what data are to be considered as measurements for adjustment: the components of deflection of the vertical $\xi$ and $\eta$, or $C_{ik}$ values from (5). Since no simple relationship (observation equation) with a measurement result on one side, and unknowns on the other side of an equation can be written, computation ought to be made under conditions of adjustment of direct measurements, rather than with measured unknowns - this is, however, excessively demanding for computation, requiring excessive storage capacity. Hence on behalf of measurements, two approximations can be applied: i) geoid heights are left uncorrected - thus, they are input to adjustment as constraints, ii) $C_{ik}$ on the left hand side of fundamental equation (5) are considered as fictitious measurements and corrected. Thereby observation equation (4) becomes:**

$$C_{ik} + v_{ik} = N_k - N_i \tag{6}$$

**permitting computation under conditions given by adjusting indirect measurements between unknowns. The first approximation is justified since reliability of given $N$ values exceeds that of the computed values considerably (a principle applied also to geodetic control networks). Validity of the second approximation will be addressed later in connection with the problem of weighting. For every triangle side of the interpolated net, an observation equation based on Eq. (6):**

$$v_{ik} = N_k - N_i - C_{ik}$$

**may be written. In matrix form:**

$$\underset{(m,1)}{\mathbf{v}} = \underset{(m,n)}{\mathbf{A}} \underset{(n,1)}{\mathbf{x}} + \underset{(m,1)}{\mathbf{l}}$$

**where A is the coefficient matrix of observation equations, x is the vector of unknowns $N$, l is the vector of constant terms; $m$ is the number of sides in the interpolation net; and $n$ is the number of points.**

## 3. THE COLLOCATION SOLUTION

**The collocation method has successfully been used for recovering gravity anomalies and geoid heights using torsion balance measurements in various test areas of Hungary (Tóth et al, 2002a,b).**

## 3.1 Detrending of gradients

**First, geodetic coordinates of all the 249 points were computed in the WGS-84 geocentric system from plane coordinates in the Hungarian Unified National Projection System (EOV). This was required for the GRAVSOFT software used for collocation (Tschering, 1994). Second, residual gravity gradients were created by a two-step process. In the first step the following normal effects on the gradients have been removed**

$$U_{xz} = \frac{1}{M}\gamma_e \beta \sin 2\varphi \cong 8.12 \sin 2\varphi \qquad \text{[E.U.]} \tag{7}$$

$$U_{yy} - U_{xx} = \gamma\left(\frac{1}{M} - \frac{1}{N}\right) \cong 10.26 \cos^2 \varphi \qquad \text{[E.U.]}\,, \tag{8}$$

**where $\gamma_e$ is the normal gravity at the equator, $\beta$ denotes gravity flattening and $M$ and $N$ are curvatures in the meridian and prime vertical directions, respectively at the point with latitude $\varphi$ and normal gravity $\gamma$.**

**Table 1. Statistics of residual gravity gradients in E.U. after removing the normal effect and a linear trend from 249 torsion balance measurements.**

| measurement | mean | min. | max. | std.dev. |
|---|---|---|---|---|
| $W_{xz}$ | 0.74 | -24.8 | 31.8 | ±10.0 |
| $W_{yz}$ | -0.81 | -33.7 | 33.1 | ±11.3 |
| $2W_{xy}$ | 1.38 | -65.6 | 42.5 | ±14.4 |
| $W_{\Delta}$ | -3.53 | -69.5 | 43.8 | ±13.1 |

**In the second step a local linear $f_1(\varphi,\lambda)$ trend has been removed from the residual gradients**

$$f_1(\varphi,\lambda) = c_1 + c_2(\varphi - \varphi_0) + c_3(\lambda - \lambda_0)\,. \tag{9}$$

**In this equation $c_1$, $c_2$, $c_3$ and $\varphi_0$, $\lambda_0$ are constants, whereas $\varphi$ and $\lambda$ are geodetic latitude and longitude of points, respectively. This procedure was similar to the one used by Hein and Jochemczyk (1979) in Germany when modeling local covariance functions from torsion balance gradients. Statistics of the residuals can be found in Table 1.**

## 3.2 Covariance function determination

**Two empirical covariance functions were determined for the selected area from trend-reduced gravity gradients. The first determination was based on the mixed horizontal-vertical combination (gradient combination) $\{W_{xz},\ W_{yz}\}$, while the second one was based on the pure horizontal combination of gradients (curvature combination)**

$\{2W_{xy},\ \ W_{yy}-W_{xx}\}$. **The next step was to approximate these empirical functions by a covariance model. The Tscherning-Rapp model 2 with *B*=4 in the denominator was chosen for this purpose (Tscherning, 1994). Therefore, the degree variances of the disturbing potential were**

$$\sigma_\ell^2(T)=\frac{A}{(\ell-1)(\ell-2)(\ell+4)} \qquad \ell \geq 3\,. \tag{10}$$

**The empirical and model covariance functions are shown in Fig. 1.**

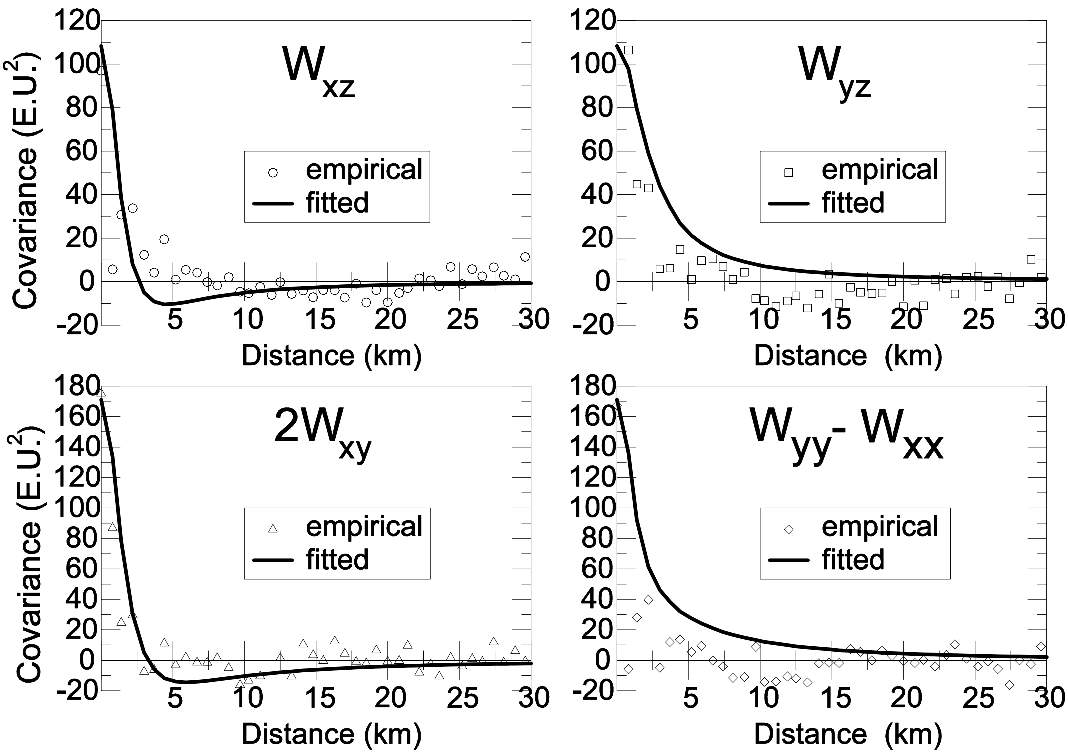


**Fig. 1. Autocovariance functions for the test area computed from gradients or curvatures. A linear trend was removed from the residual gradients.**

**The collocation formula is given by e.g. Moritz (1980)**

$$N(P)=C^{NW_{aa}}(\psi_{Pi})(C^{W_{aa}W_{aa}}(\psi_{ii'}))^{-1}\Delta W_{aa}\,. \tag{11}$$

$C^{W_{aa}W_{aa}}(\psi)$ **is the auto covariance function of the** $\Delta W_{aa}$ **residual gradients (the index *aa* denotes either the combination of gradients or curvatures),** $C^{NW_{aa}}(\psi)$ **is the cross covariance function of geoid heights and residual gradients. Similar formulas apply if the** $(\xi\,,\eta)$ **deflections of the vertical are predicted instead of the geoid heights *N*. These** formulas were used with the above determined model covariance functions to predict various gravity field quantities for the test area, namely geoid heights and deflections of the vertical.

## 4. TEST COMPUTATIONS

Test computations were performed in a Hungarian area extending over about 800 $km^2$. There were 249 torsion balance stations, and 13 points (3 astrogeodetic, and 10 astrogravimetric points) among them where $\xi$, $\eta$ and $N$ values were known in this test area referring to the *GRS80* system. The 3 astrogeodetic points indicated with squares in Fig. 2 were used as initial (fixed) points of interpolations and the 10 astrogravimetric points indicated with triangles in Fig. 2 were used for checking of computations.

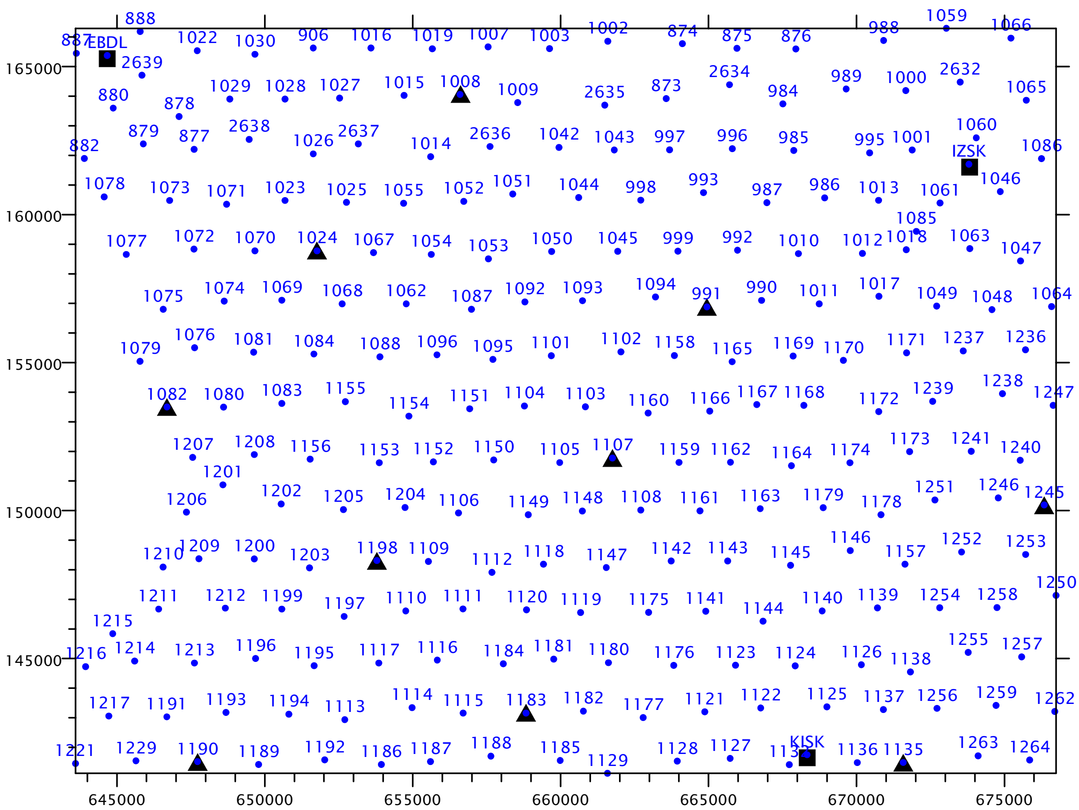


**Fig. 2 The test area. Coordinates are in meters in the Hungarian Unified National Projections (EOV) system.**

The interpolation network in Fig. 2 has 249 points in all and 246 of these are points with unknown deflections. Since there are two unknown components of deflection of the vertical at each point there are 498 unknowns for which 683 equations can be written.

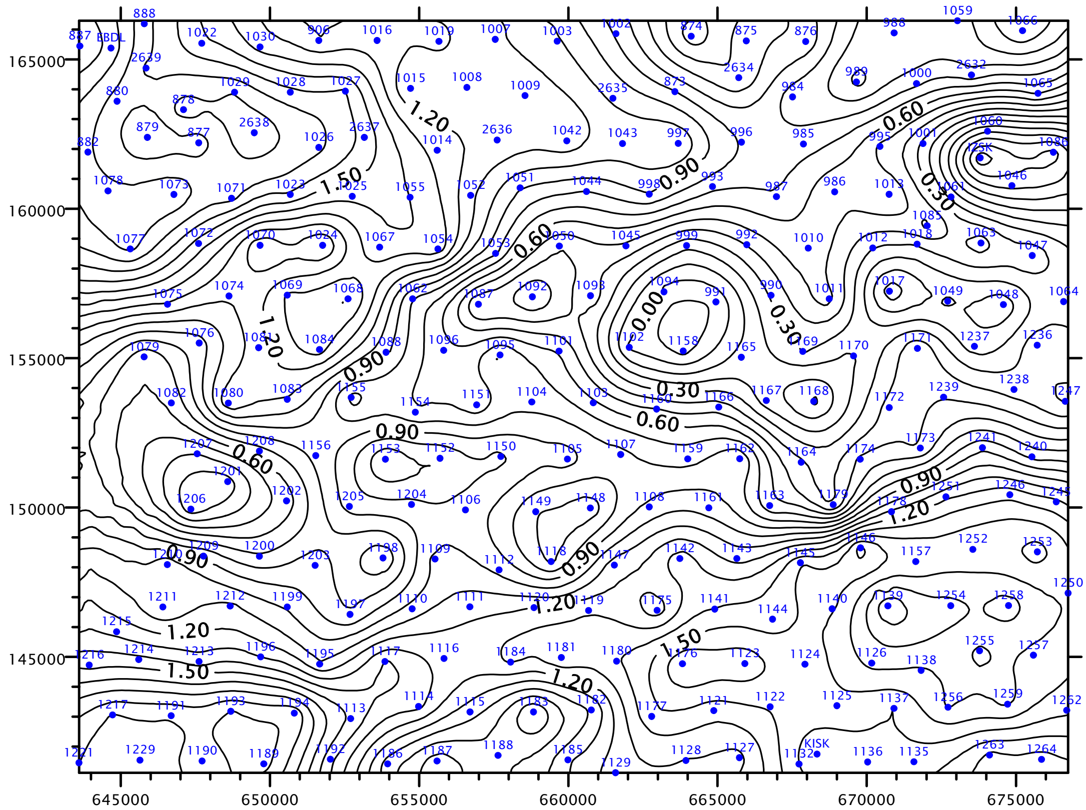


**Fig. 3 Computed ξ component from collocation. Isoline interval is 0.1”**

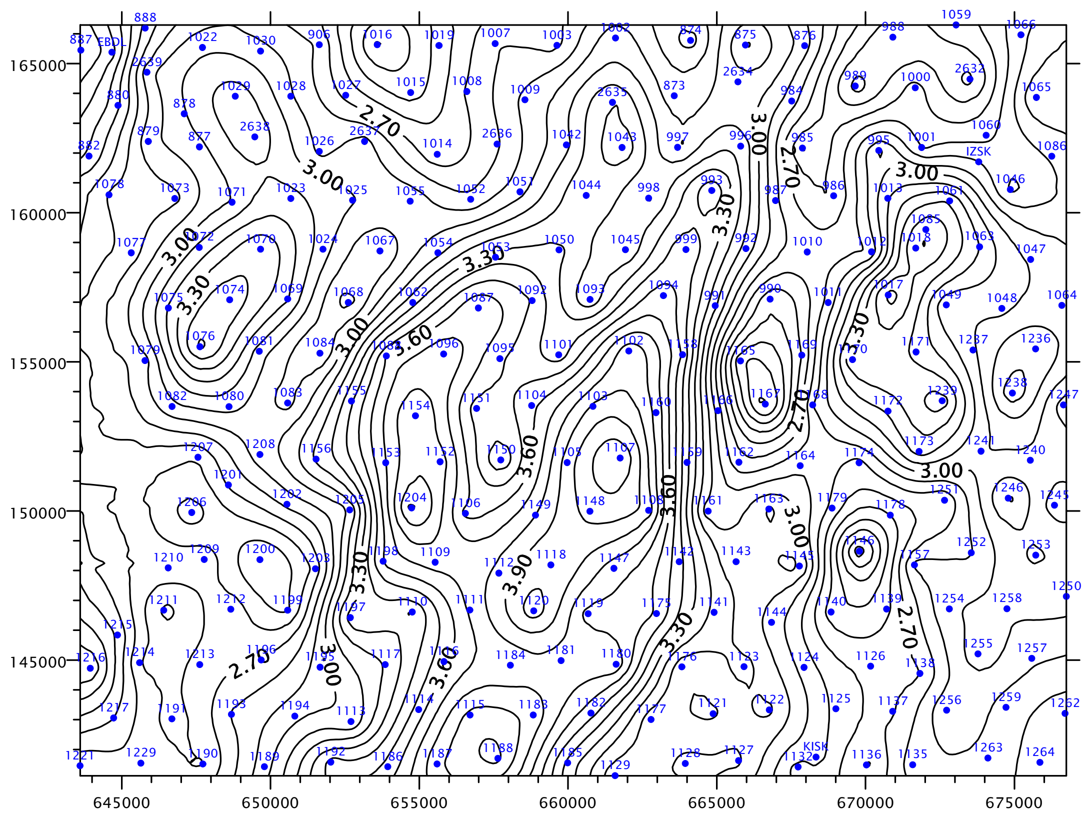


**Fig. 4 Computed η component from collocation. Isoline interval is 0.1”**

**In Figs. 3 and 4 $\xi$ and $\eta$ components of deflections of the vertical are visualized in isoline maps that resulted from the collocation solution.**

**Based on the previously computed deflection of the vertical components, geoid computations were carried out. The computed geoid map can be seen on Fig. 5.**

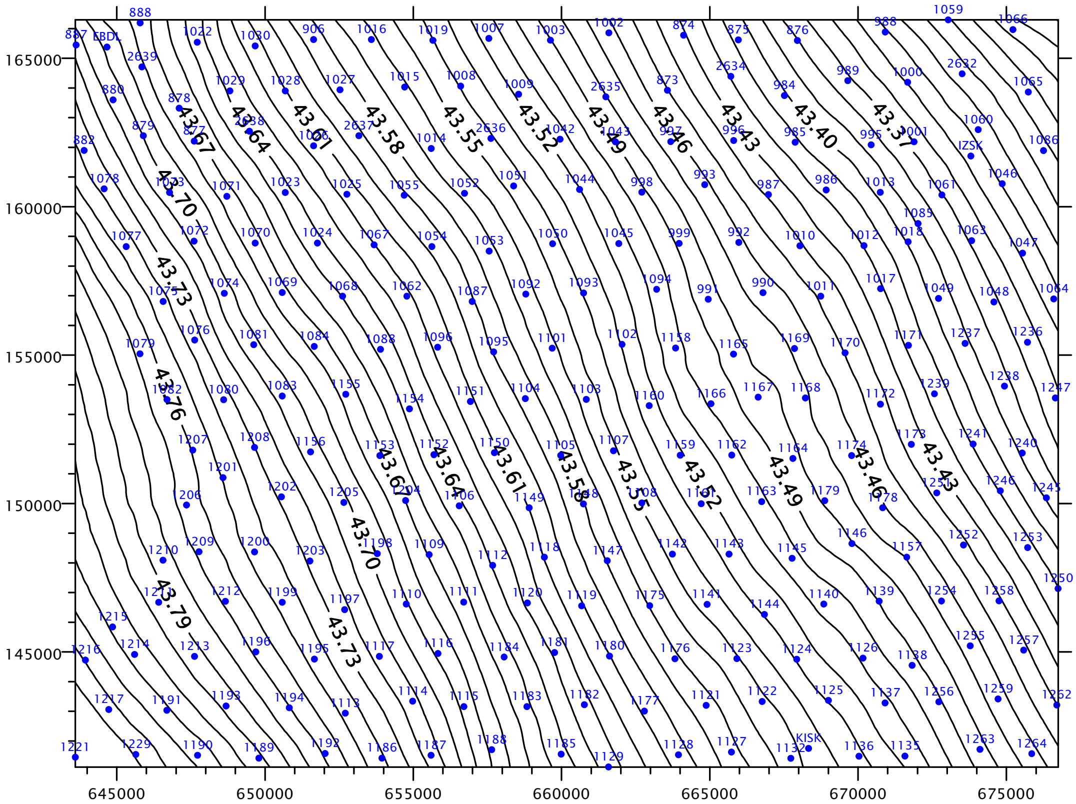


**Fig. 5 Geoid heights from collocation ($W_\Delta$ , $2W_{xy}$ solution). Contor interval is 0.01m.**

**In order to be conformant with the interpolation solution, we have used for the numerical tests the same 3 astrogeodetic fixed points (Fig. 2) for the collocation. This was achieved by assigning very large weights (small standard deviations) to these three stations. (The uniform standard deviation of residual gravity gradients during the prediction step was assigned as ±2 E.U., while these fixed points were assigned the standard deviations of ±0.0001 m or ±0.001" in case of geoid height or deflection predictions, respectively). Of course, for correct computations, the geoid heights or the deflections in these control points have to be of zero mean, i.e. the mean values have to be removed before the collocation step.**

**An independent gravimetric geoid solution for Hungary, the HGTUB2000 solution, based on gravity anomalies was used for the evaluation of our predictions with interpolation and collocation (Tóth and Rózsa, 2000). Table 2 shows the differences of geoid heights at the 10 checkpoints.**

**The statistics of the geoid height and vertical deflection differences for the collocation and interpolation methods are presented in Table 3. These statistics shows that the pure horizontal gradients, i.e. the $W_{xx}$-$W_{yy}$, $2W_{xy}$ in this area at the 10 checkpoints yielded a better fit to both the gravimetric geoid undulations and astronomical deflection of the vertical. Therefore no collocation solution was provided with all the four gradients $W_\Delta$ ,**

$2W_{xy}$ , $W_{zx}$ , $W_{zy}$ . **The $\eta$ component fits better than the $\xi$ component in both collocation solution with the interpolated deflections.**

**Table 2. Geoid height differences with reference gravimetic geoid heights at 10 checkpoints for the interpolation and collocation method [mm].**

| Checkpoint No. | $N_{grav}$-$N_{int}$ | $N_{grav}$-$N_{coll}\{W_{xz},W_{yz}\}$ | $N_{grav}$-$N_{coll}\{2W_{xy},W_{\Delta}\}$ |
|---|---|---|---|
| 991 | 27 | 3 | -8 |
| 1008 | 102 | 2 | 22 |
| 1024 | 99 | 32 | 36 |
| 1082 | 145 | 59 | 34 |
| 1107 | 32 | 15 | 0 |
| 1135 | -79 | -9 | -6 |
| 1183 | 31 | 44 | 18 |
| 1190 | 40 | 94 | 3 |
| 1198 | 104 | 58 | 27 |
| 1245 | -124 | -23 | -23 |
| std. deviation: | ±80 | ±35 | ±19 |

**Surface maps of the $\xi$ and $\eta$ components of deflections of the vertical and an isoline map of geoid undulation differences between collocation and interpolation are visualized in Figs. 6, 7 and 8 respectively. The isoline map of the geoid undulation differences shows mainly a linear trend in the Eastern part of the area, ranging from 6 to –12 cm.**

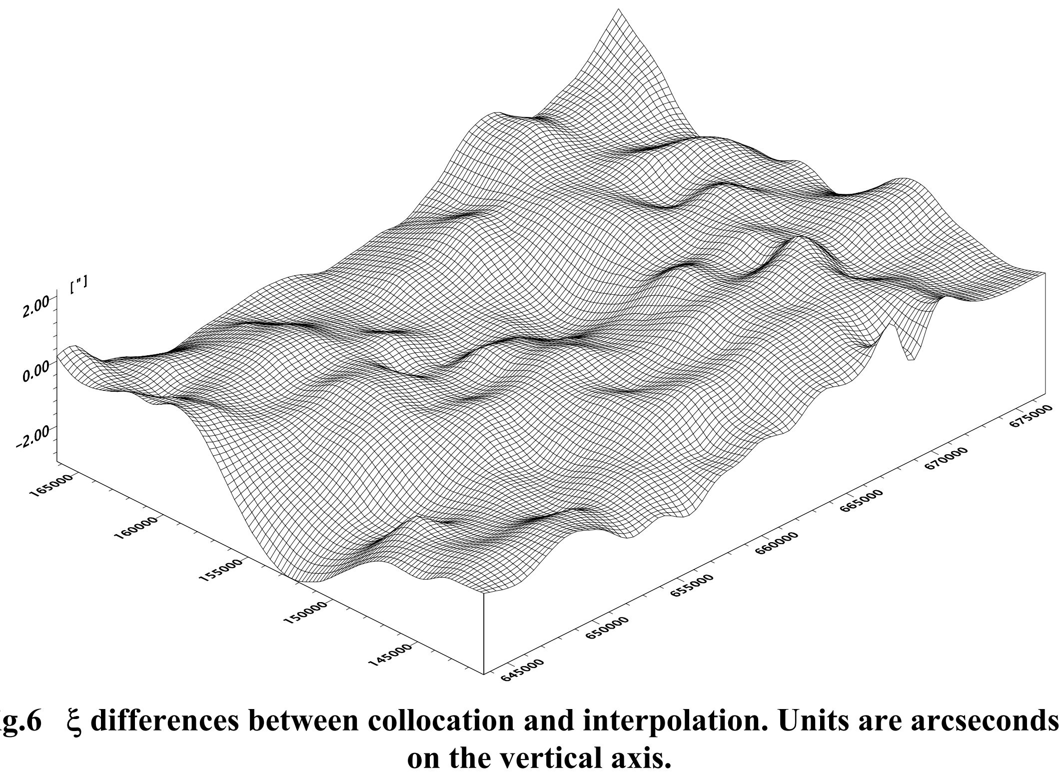


**Fig.6 ξ differences between collocation and interpolation. Units are arcseconds [“] on the vertical axis.**

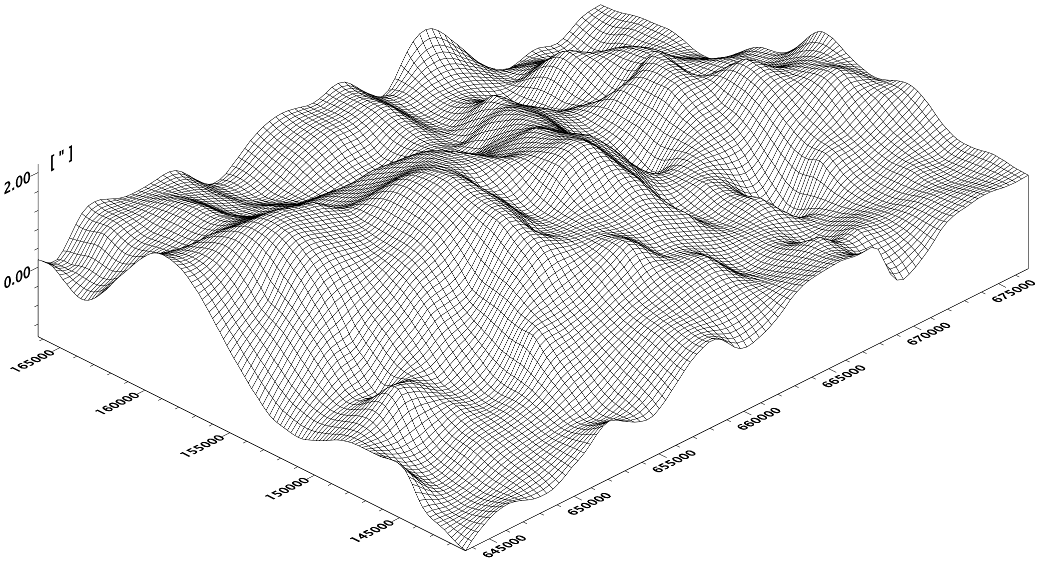


**Fig.7 η differences between collocation and interpolation. Units are arcseconds ["] on the vertical axis.**

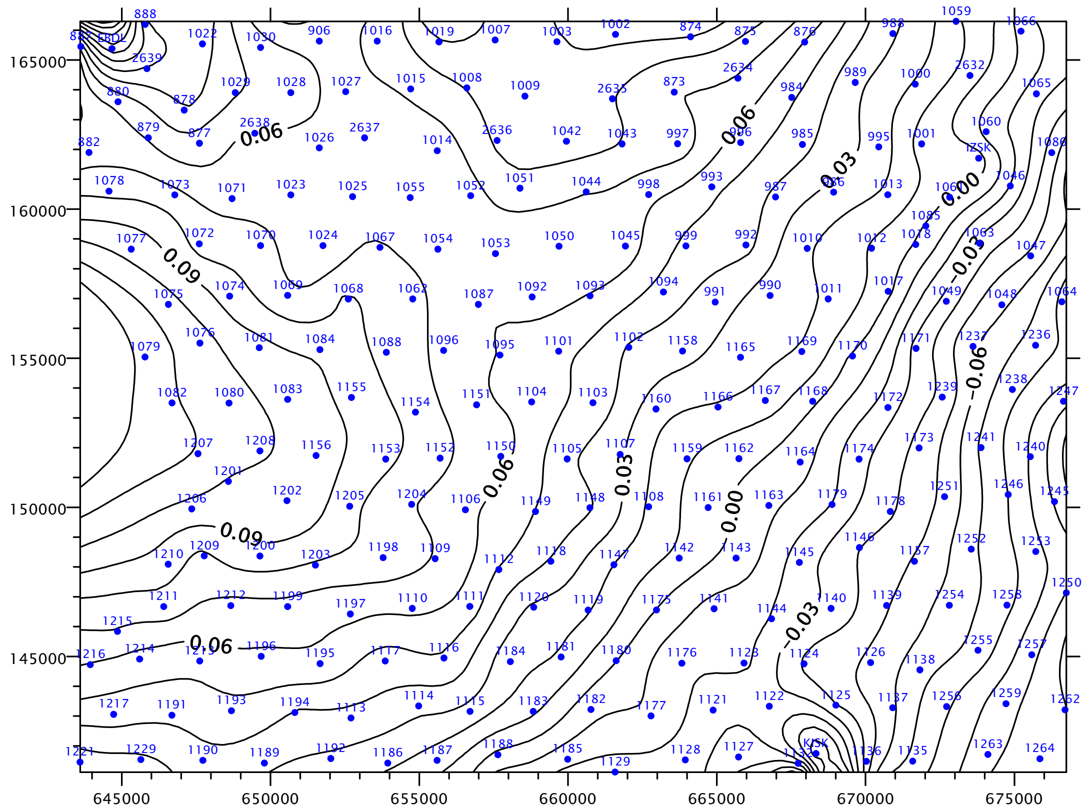


**Fig.8 N differences between collocation and interpolation. Isoline interval is 0.01m.**

**Table 3. Statistics of geoid height differences [m] and deflections of the vertical ["] at 10 checkpoints between the collocation and interpolation methods.**

| | min. | max. | mean | std.dev. |
|---|---|---|---|---|
| $N_{coll}\{W_{xz},W_{yz}\}-N_{int}$ | -0.104 | 0.114 | 0.013 | ±0.057 |
| $\xi_{coll}\{W_{xz},W_{yz}\}-\xi_{int}$ | -2.249 | 1.566 | -0.157 | ±0.875 |
| $\eta_{coll}\{W_{xz},W_{yz}\}-\eta_{int}$ | -1.768 | 2.223 | 0.512 | ±0.774 |
| $N_{coll}\{2W_{xy},W_{\Delta}\}-N_{int}$ | -0.110 | 0.117 | 0.026 | ±0.054 |
| $\xi_{coll}\{2W_{xy},W_{\Delta}\}-\xi_{int}$ | -2.034 | 1.634 | -0.144 | ±0.735 |
| $\eta_{coll}\{2W_{xy},W_{\Delta}\}-\eta_{int}$ | -1.301 | 2.207 | 0.558 | ±0.697 |

## CONCLUSIONS

**By collocation two independent solutions for deflections of the vertical and geoid heights were computed from $W_{zx}$ , $W_{zy}$ and $W_{\Delta}$ , $2W_{xy}$ gradients, using astrogeodetic data to achieve an agreement with the interpolation method. The standard deviation of geoid height differences at checkpoints were about ±1-3 cm. The $W_{\Delta}$ , $2W_{xy}$ combination yielded slightly better results since the maximum geoid height difference was only 3.6 cm. The differences in the deflection components were generally below 1", slightly better for the $\eta$ component. The geoid height differences between interpolation and collocation may be partly caused by the different treatment of the torsion balance data (trend removal). The results confirm the fact that gravity gradients give good possibility to compute very precise local geoid heights. Since it is possible to compute geoid heights and deflections of the vertical by surface integration it would be interesting to compare this method with interpolation by line integration as well.**

## ACKNOWLEDGEMENT

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