Inversion reconstruction of gravity potential based on gravity gradients

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Abstract. A new method was worked out for the inversion reconstruction of gravity potential. This method gives a possibility to determine the potential function and all of its important derivatives using the common inversion of gravity gradients and the first derivatives of potential. Gravity gradients can be originated from Torsion balance measurements, while the first derivatives of potential can be derived from the deflections of the vertical data. Different fields having great importance can be originated from this reconstructed potential function at any points of the investigated area. Advantage of this method is that the solution can be performed by a significantly overdetermined inverse problem.

Test computations were performed for the inversion reconstruction of gravity potential. There were 248 torsion balance measurements and 13 points where the deflections of the vertical are known in our test area. This inversion algorithm is rather stable. Gravity potential, the first and the second derivatives of the potential were determined for the test area by this suggested method. This method gives a good possibility for a useful geodetic application; deflections of the vertical based on torsion balance measurements can be determined for the whole area for each torsion balance stations.

Keywords. inversion, gravity potential, curvature gradients of gravity, deflection of the vertical, torsion balance measurements

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Introduction

In consequence of scientific activities of Hungarian physicist Loránd Eötvös in the 20th century more than 60000 torsion balance measurements were made in Hungary. At present time serious efforts are going on for rescuing the historical torsion balance measurements; nowadays 24544 torsion balance measurements are available for further processing in computer database.

Under Hungarian conditions, in addition to gradient values W_{zx} and W_{zy}, also curvature data W_{xy} and $W_{\Delta} = W_{yy} - W_{xx}$ are available with great precision. Since earlier torsion balance measurements were made mainly for geophysical prospecting, mostly only gravity gradients have been processed; up to now, gravity curvature values have been left unprocessed.

New computer technology opened new vistas on the area of geodetic applications of torsion balance measurements. Earlier researches have started, first about interpolation of deflection of the vertical based on torsion balance measurements over larger regions (Völgyesi 1993, 1995, 2005), then for the determination of the fine details of the geoid based on deflections of the vertical (Völgyesi 1998, 2001; Tóth and Völgyesi 2002).

A new method for inversion reconstruction of gravity potential suggested here. This method gives a new possibility to determine deflections of the vertical based on torsion balance measurements.

1 Inversion algorithm

Let us choose the gravity potential $W(x,y)$ as an expansion in a series of a known set of basis function $\Psi_0 .... \Psi_P$:

$$W(x,y) = \sum_{n=1}^{P+1} \sum_{i=0}^{n-1} B_j \Psi_i(y) \Psi_l(x)$$

where $j = \dfrac{n(n-1)}{2} + i,\ l = n - i - 1$ and B_j are unknown coefficients of the expansion in a series. The number of unknowns are $M = \dfrac{P(P+1)}{2} + P + 1$. Generally, there are different types of basis functions, - in our case power functions, or Chebishev, Legendre, … etc polynomials are supposed as basis functions.

So the solution of the direct problem (the values of curvature data W_Δ and W_{xy}) can be computed as

$$^{comp}W_\Delta = \frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2} = \sum_{n=1}^{P+1}\sum_{i=0}^{n-1} B_j \left\{ \Psi_i''(y)\Psi_l(x) - \Psi_i(y)\Psi''_l(x) \right\}$$

and

$$^{comp}W_{xy} = \frac{\partial^2 W}{\partial x \partial y} = \sum_{n=1}^{P+1}\sum_{i=0}^{n-1} B_j \Psi_i'(y)\Psi'_l(x) \ .$$

Introducing the notations

$$Q_{kj} = \Psi_i''(y_k)\Psi_l(x_k) - \Psi_i(y_k)\Psi''_l(x_k)$$

and

$$S_{kj} = \Psi_i'(y_k)\Psi'_l(x_k)$$

the computed torsion balance data in an arbitrary point $P_k(x_k, y_k)$ are

$$^{comp}W_\Delta^{(k)} = \sum_{j=1}^{M} B_j Q_{kj} \quad \text{and} \quad ^{comp}W_{xy}^{(k)} = \sum_{j=1}^{M} B_j S_{kj} \ ,$$

where S_{kj}, Q_{kj} are known matrix elements (the index j is determined uniquely by i and l).

So the discrepancies of the measured and the computed data are

$$e_k = {}^{meas}W_{xy}^{(k)} - \sum_{j=1}^{M} B_j S_{kj}$$

and

$$f_k = {}^{meas}W_\Delta^{(k)} - \sum_{j=1}^{M} B_j Q_{kj} \ .$$

Let the function have to be minimized the norm L_2 of the discrepancy vector:

$$E = \sum_{k=1}^{N_1} e_k^2 + \sum_{k=1}^{N_2} f_k^2$$

where N_1 and N_2 are the numbers of measured data W_{xy} and W_Δ respectively. Let us introduce the notations

$$^{meas}\vec{d} = \left\{ W_{xy}^{(1)}, ..., W_{xy}^{(N_1)}, W_\Delta^{(1)}, ..., W_\Delta^{(N_2)} \right\},$$

$$G_{kj} = \begin{cases} S_{kj} & k \le N_1 \\ Q_{kj} & N_1 < k \le N_1 + N_2 \end{cases}$$

and the

$$\vec{e} = {}^{meas}\vec{d} - \underline{\underline{G}}\vec{B} \ ,$$

$$E = (\vec{e}, \vec{e}) = \sum_{k=1}^{N} e_k^2$$

for the vectorial discussion, where $N = N_1 + N_2$. Solution of this inverse problem is based on the condition system $\frac{\partial E}{\partial B_l} = 0$, $(l = 1, ..., M)$ resulting in the set of normal equations

$$\underline{\underline{G}}^T \underline{\underline{G}} \vec{B} = \underline{\underline{G}}^T \ {}^{meas}\vec{d} \ .$$

So this inverse problem is linear, vector $\vec{B}$ of expansion in a series' coefficients can be determined by solving the above set of equations

$$\vec{B} = \left(\underline{\underline{G}}^T \underline{\underline{G}} \right)^{-1} \underline{\underline{G}}^T \vec{d} \ .$$

Unfortunately there are no equations for the coefficients of B_0, B_1, B_2 (constant term, and the coefficients of linear terms of *x* and *y*), because of torsion balance can measure only the second derivates of gravity potential.

Further on, the constant term B_0 is not important for us because this is only an additive constant for the potential. But determination of the B_1 and B_2 are important, which requires further independent information.

This information may come from deflection of the vertical's data.

The two components of deflections of the vertical in *N-S* and *E-W* directions:

$$\xi = \Phi - \varphi, \ \text{and} \ \eta = (\Lambda - \lambda)\cos\varphi$$

where Φ, Λ are the astronomical and φ, λ are the geographical latitude and longitude respectively. At the same time

$$\xi = \frac{W_x}{g} - \frac{U_x}{\gamma} \quad \text{and} \quad \eta = \frac{W_y}{g} - \frac{U_y}{\gamma}$$

where *U* is the potential of normal gravity field, *g* is the real and γ is the normal gravity (Völgyesi, 2005). The first derivatives from the deflection of the vertical's components:

$$W_x = g\xi + \frac{g}{\gamma}U_x \quad \text{and} \quad W_y = g\eta + \frac{g}{\gamma}U_y \,.$$

In a suitable local coordinate system $U_x \approx U_y \approx 0$, therefore

$$W_x = g\xi \quad \text{and} \quad W_y = g\eta \,.$$

From the expansion in a series of gravity potential:

$$W_x = \frac{\partial W}{\partial x} = \sum_{n=1}^{P+1}\sum_{i=0}^{n-1} B_j \Psi_i^{'}(x)\, \Psi_l(y) = B_1 + F(x,y)$$

$$W_y = \frac{\partial W}{\partial y} = \sum_{n=1}^{P+1}\sum_{i=0}^{n-1} B_j \Psi_i(x)\, \Psi^{'}{}_l(y) = B_2 + G(x,y)$$

where $F(x,y)$, $G(x,y)$ are known, because the coefficients $B_j\ (j \ge 3)$ are given. E.g. if deflection of the vertical components (ξ_1, η_1) are known at point $P_1(x = x_1,\ y = y_1)$, then

$$B_1 = -F(x_1, y_1) + g\xi_1$$

and

$$B_2 = -G(x_1, y_1) + g\eta_1 \,.$$

So the potential function - apart from an additive constant - can be determined at any points of the region covered by torsion balance measurements using the coefficients of expansion in a series of a known set of basis function.

Unfortunately this algorithm can not be used at any times in this form, because the coefficients of the power series may not be independent from each other, if we use power functions or Chebishev, Legendre, etc. polynomials as basis functions (Dobróka, Völgyesi 2005). E.g. according to our investigations there are all constant terms in the first three columns of the Jacobi matrix $\underline{\underline{G}}$ referring to every torsion balance points (Dobróka, Völgyesi 2005). So this three columns are not linearly independent therefore the matrix $\underline{\underline{G}}^T \underline{\underline{G}}$ of normal equations will be singular, and the inverse problem can not be solvable.

The joint inversion of the data arising from various physical background is well-known in geophysical inversion (Dobroka et al., 1991). Our present problem is similar and can be handled on the same way: by integrating the torsion balance data and deflection of the vertical data into a single (joint) inversion procedure.

Data of first derivatives can be originated from the deflections of the vertical:

$$^{meas}W_x = {}^{meas}\xi\, g \quad \text{and} \quad {}^{meas}W_y = {}^{meas}\eta\, g \,.$$

The theoretical values of these data can be determined by using the expansion formula of potential:

$$^{comp}W_x = \frac{\partial W}{\partial x} = \sum_{n=1}^{P+1}\sum_{i=0}^{n-1} B_j \Psi_i^{'}(x)\Psi\ _l(y) = \sum_{j=1}^{M} B_j R_{kj}$$

$$^{comp}W_y = \frac{\partial W}{\partial y} = \sum_{n=1}^{P+1}\sum_{i=0}^{n-1} B_j \Psi_i\ (x)\Psi^{'}{}_l(y) = \sum_{j=1}^{M} B_j Z_{kj}$$

where

$$R_{kj} = \Psi_i^{'}(x_k)\Psi\ _l(y_k)\,,$$

$$Z_{kj} = \Psi_i\ (x_k)\Psi^{'}{}_l(y_k)\,,$$

$$(j = \frac{n(n-1)}{2} + i)\,.$$

Integrating the vector of torsion balance data and the first derivatives data derived from deflections of the vertical into a complete data vector we define:

$$^{meas}\vec{d} = \left\{ \begin{array}{c} W_{xy}^{(1)},\ldots,W_{xy}^{(N_1)},W_{\Delta}^{(1)},\ldots,W_{\Delta}^{(N_2)}, \\ W_x^{(1)},\ldots,W_x^{(N_3)},W_y^{(1)},\ldots,W_y^{(N_4)} \end{array} \right\}.$$

Vector of computed data can be prepared in a similar form, but extension of structure of Jacobi matrix is necessary:

$$G_{kj} = \begin{cases} S_{kj} & k \le N_1 \\ Q_{kj} & N_1 < k \le N_1 + N_2 \\ R_{kj} & N_1 + N_2 < k \le N_1 + N_2 + N_3 \\ Z_{kj} & N_1 + N_2 + N_3 < k \le N_1 + N_2 + N_3 + N_4 \end{cases}$$

So, the computed data:

$$^{comp}d_k = \sum_{j=1}^{M} B_j G_{kj} \quad \text{or} \quad {}^{comp}\vec{d} = \underline{\underline{G}}\vec{B}$$

and discrepancies of the measured and the computed data:

$$\vec{e} = {}^{meas}\vec{d} - {}^{comp}\vec{d} = {}^{meas}\vec{d} - \underline{\underline{G}}\vec{B}\,,$$

$$E = (\vec{e},\vec{e}) = \sum_{k=1}^{N} e_k^2\,,$$

$$N = N_1 + N_2 + N_3 + N_4 \, .$$

Applying $\dfrac{\partial E}{\partial B_l} = 0$, $(l = 1,\ldots,M)$, solution of this inverse problem can get too:

$$\underline{\underline{G}}^T \underline{\underline{G}} \vec{B} = \underline{\underline{G}}^T \; {}^{meas}\vec{d}$$

where from:

$$\vec{B} = \left(\underline{\underline{G}}^T \underline{\underline{G}} \right)^{-1} \underline{\underline{G}}^T \vec{d} \; .$$

This inversion algorithm is rather stable as it was proved earlier by test computations (Dobróka, Völgyesi 2005).

2 Test computations

Test computations were performed in an area extending over some 750 km^2 where 248 torsion balance stations can be found. There are 13 from these torsion balance points where astrogeodetic or astrogravimetric data are available too. Both topographic conditions, density of torsion balance measurements and astrogeodetic stations reflects average conditions of Hungary here. Location of torsion balance stations can be seen on Fig. 1 marked by small circles. The 3 astrogeodetic points indicated with squares and the 10 astrogravimetric points indicated with triangles on Fig. 1 were used as initial (fixed) points of computations. ξ and η values were known in these points referring to the GRS80 system.

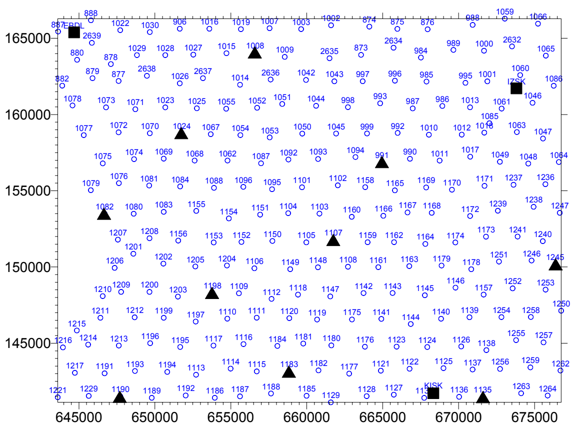

Fig. 1. The test area (coordinates are in meters in the Hungarian Unified National Projections (EOV) system).

In Figs. 2 and 3 curvature gradients W_Δ and $2W_{xy}$ measured by torsion balance are visualized on our test area. The figures show relatively high spatial variations predicting the need of high polinomial order in the series expansion represenation of the potential field given above.

Using the W_Δ and $2W_{xy}$ data sets as well as the W_x and W_y data derived from astrogravimetric and astrogeodetic data we made some tests of the joint inversion algorithm proposed in this paper.

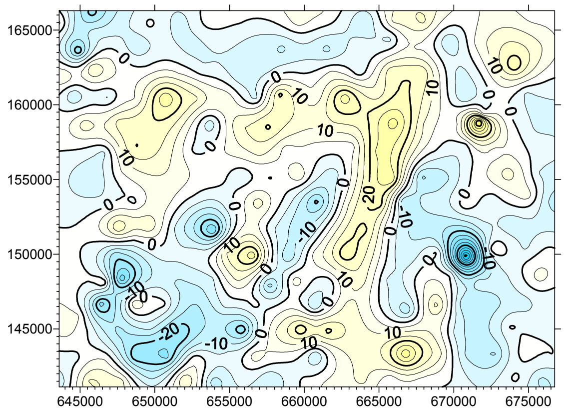

Fig. 2. Isoline map of curvature data W_Δ on the test area. Isoline interval is 5 E. (1E = 1Eötvös Unit = 10^{-9} s^{-2})

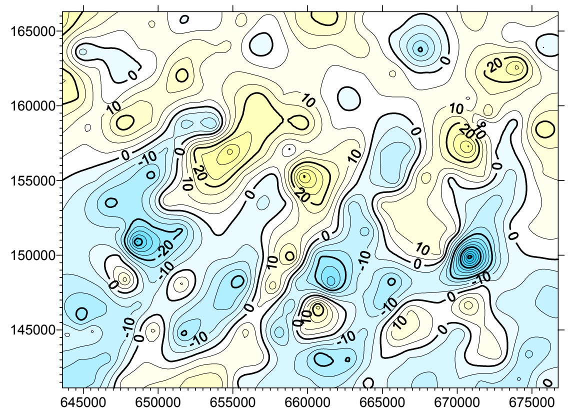

Fig.3. Isoline map of curvature data $2W_{xy}$ on the test area. Isoline interval is 5 E. (1E = 1Eötvös Unit = 10^{-9} s^{-2})

Solving the inverse problem the expansion coefficients are determined with the help of which the potential field and both of its first and second derivatives (including W_Δ and $2W_{xy}$) can be found at any point of the test area. Fig. 4 Fig. 5 show the isoline map of the calculated (predicted) W_Δ and $2W_{xy}$ data. The polynomial order was *P=30* in this case which means 496 unknowns. The fit between the measured data (Figs. 2 and 3) and the predictions (Figs. 4 and 5) seems to be satisfactory.

Our experiences show that care should be taken in choosing the polynomial order because increasing its value the condition number of the normal equation increases rapidly. This can make the parameter estimation (coefficients *B*) unreliable with high estimation errors and strong correlation between some coefficients. It was found, that P=18 – 24 can give good compromise between resolution and stability in our problem.

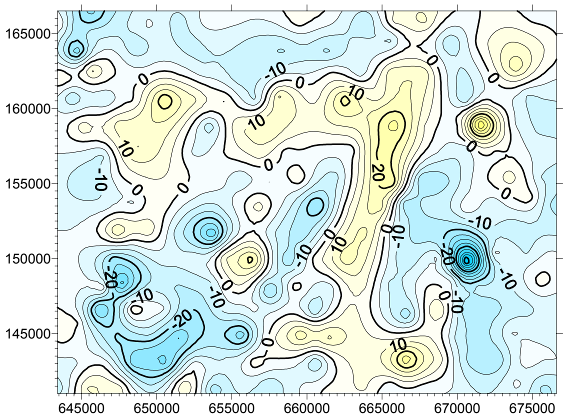

Fig. 4. Computed curvature field W_{Δ} from joint inversion. Isoline interval is 5 E. (1E = 1Eötvös Unit = 10^{-9} s^{-2})

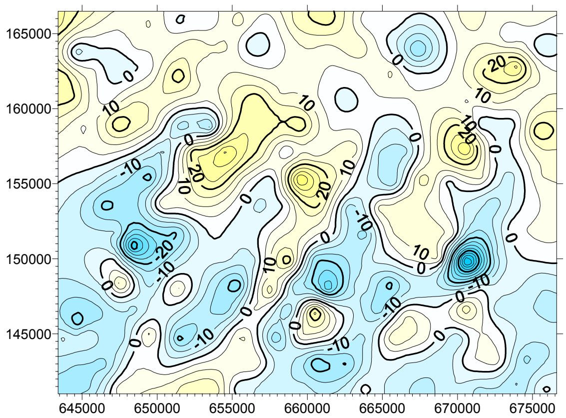

Fig.5. Computed curvature field $2W_{xy}$ from joint inversion. Isoline interval is 5 E. (1E = 1Eötvös Unit = 10^{-9} s^{-2})

In the knowledge of the expansion coefficients, it is also possible to compute the potential field and its first derivatives by using the expansion formula. Apart from an additive constant gravity potential field is shown by Fig. 6 as an isoline map (isoline interval is 0.1 m^2s^{-2}). Figs. 7 and 8 show the isoline map of the first derivatives W_x and W_y. The polynomial order was chosen *P=19*, in this case. Isoline interval is 0.5 mGal (1 mGal = 10^{-5} ms^{-2}) on these figures.

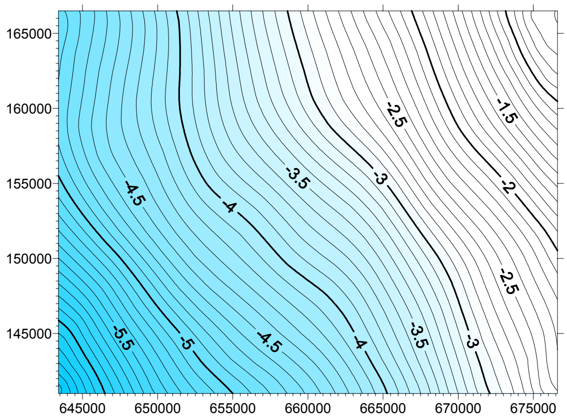

Fig. 6. Computed potential field from joint inversion. Isoline interval is 0.1 m^2s^{-2}

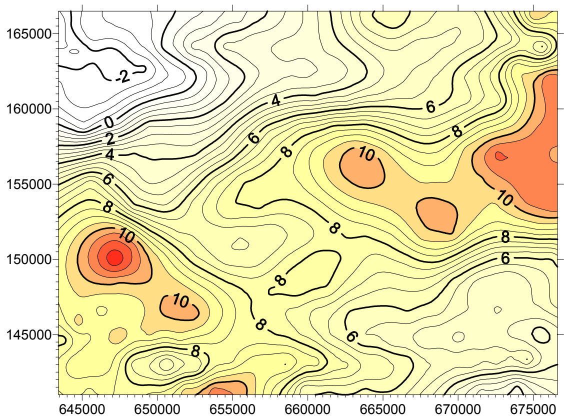

Fig. 7. Computed W_x field from joint inversion. Isoline interval is 0.5 mGal (1 mGal = 10^{-5} ms^{-2})

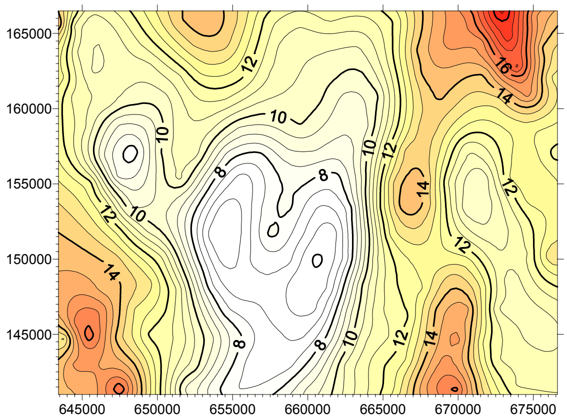

Fig. 8. Computed W_y field from joint inversion. Isoline interval is 0.5 mGal (1 mGal = 10^{-5} ms^{-2})

3 Geodetic applications

Using equations $\xi = W_x / g$ and $\eta = W_y / g$ components of deflection of the vertical can be determined from the first derivatives of potential W_x and W_y. Fig. 9 shows the vector map of deflections of the vertical on the test area. (Deflections of the vertical can be imagined as vectors, the lengths of vectors are $\theta = \sqrt{\xi^2 + \eta^2}$ and the positive direction of vectors shows from the ellipsoidal zenith to the astronomical one.)

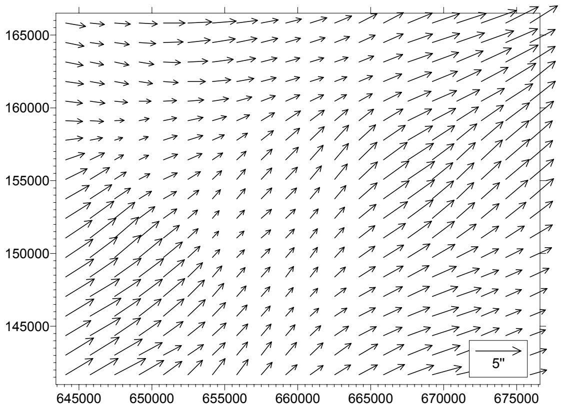

Fig. 9. Computed vectors of deflections of the vertical.

As a comparison the same fields determined by using the method of collocation are shown in Fig. 10 (Tóth, Völgyesi 2002). It can be seen, that the two methods give similar results.

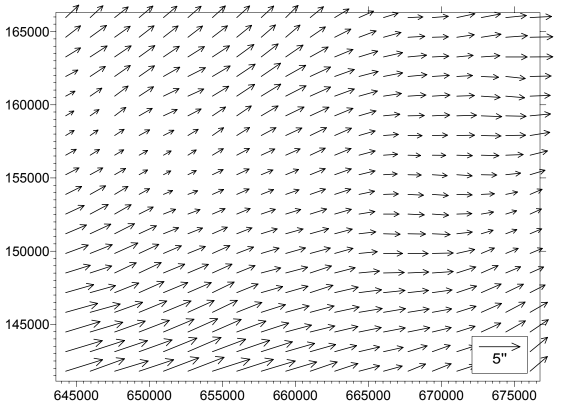

Fig. 10 Vectors of deflections of the vertical from collocation.

4 Conclusion

Our discussed method gives good a possibility for the determination of potential function by common inversion using both a large number of torsion balance and a few astronomical (deflection of the vertical) data. Different fields (the first and the second derivatives of the potential) having great importance can be originated from this reconstructed potential function at any points of the investigated area. Advantage of this method is that the solution can be performed by a significantly overdetermined inverse problem.

A useful geodetic application of this method is the determination of deflections of the vertical based on torsion balance measurements.

Acknowledgements

Our investigations are supported by the National Scientific Research Fund (OTKA T-037929).

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