Determination of gravity field from horizontal gradients of gravity

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Abstract. Torsion balance measurements are a very useful source to study the short wavelength features of the local gravity field, especially below 30 km wavelength. Our purpose is to use the gravity gradients measured by torsion balance in combination with gravity values for the determination of a more detailed and better gravity field. Previously determination of gravity anomalies was investigated (Völgyesi, Tóth, Csapó 2004), and now the determination of gravity field from torsion balance measurements is discussed.
A method was developed, based on integration of horizontal gradients of gravity W_{zx} and W_{zy} to predict gravity at all points of a torsion balance network. Test computations were performed in a characteristic flat area where both torsion balance and gravimetric measurements are available. There were 248 torsion balance stations and 1197 gravity measurements on this area. 18 points from these 248 torsion balance stations were chosen as fixed points where gravity are known from measurements and the unknown gravity values were interpolated on the remaining 230 points.

Comparison of the measured and the interpolated gravity values indicates that horizontal gradients of gravity give a possibility to determine gravity values from torsion balance measurements by *mGal* accuracy on flat areas.

Keywords. Determination of gravity field, horizontal gradients of gravity, torsion balance measurements.

1 The proposed method

Let's start from the relationship between gravity and gravity potential:

$$\mathbf{g} = -grad\, W \,, \tag{1}$$

where W is the gravity potential. Changing of gravity g between two arbitrary points P_i and P_k is:

$$\left(g_k - g_i\right) = -\left[\left(\frac{\partial W}{\partial r}\right)_k - \left(\frac{\partial W}{\partial r}\right)_i\right] .$$

In a special coordinate system (x points to North, y to East and z to Down) the changing of gravity:

$$\left(g_k - g_i\right) = \left[\left(\frac{\partial W}{\partial z}\right)_k - \left(\frac{\partial W}{\partial z}\right)_i\right] .$$

Applying the notation $W_z = \partial W / \partial z$ for the partial derivatives, the changing of gravity between the two points P_i and P_k is:

$$\left(g_k - g_i\right) = (W_z)_k - (W_z)_i \,.$$

So in the case of displacement vector **dr** the infinitesimal change of gravity g will be:

$$dg = \nabla(g)\cdot \mathbf{dr} = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz = W_{zx}dx + W_{zy}dy + W_{zz}dz \,.$$

Integrating this equation between points P_i and P_k we get the changing of gravity:

$$\left(g_k - g_i\right) = \int_i^k dg = \int_i^k W_{zx}dx + \int_i^k W_{zy}dy + \int_i^k W_{zz}dz \,, \tag{2}$$

where W_{zx} and W_{zy} are horizontal gradients of gravity measured by torsion balance, W_{zz} is the measured vertical gradient.

Let's compute the first integral on the right side of equation (2) between the points P_i and P_k. Before the integration a relocation to a new coordinate system is necessary; the connection between the coordinate systems (x, y) and the new one (u, v) can be seen on Figure 1. Denote the direction between the points P_i and P_k with *u* and be the coordinate axis *v* perpendicular to *u*. Denote the azimuth of *u* with α_{ik} and point the *z* axis to down, perpendicularly to the plane of (*xy*) and (*uv*)!

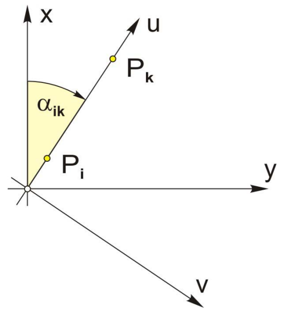

Fig. 1 Coordinate transformation $(x,y)\rightarrow(u,v)$

The transformation between the two systems is:

$$\left.\begin{aligned} x &= u\cos\alpha_{ik} - v\sin\alpha_{ik} \\ y &= u\sin\alpha_{ik} + v\cos\alpha_{ik} \end{aligned}\right\} .$$

Using these equations, the first derivatives of any function *W* are:

$$\frac{\partial W}{\partial u} = \frac{\partial W}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial W}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial W}{\partial x}\cos\alpha_{ik} + \frac{\partial W}{\partial y}\sin\alpha_{ik}$$

$$\frac{\partial W}{\partial v} = \frac{\partial W}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial W}{\partial y}\frac{\partial y}{\partial v} = -\frac{\partial W}{\partial x}\sin\alpha_{ik} + \frac{\partial W}{\partial y}\cos\alpha_{ik}$$

From this first equation

$$\begin{aligned} W_{zu}du &= \left(W_{zx}\cos\alpha_{ik} + W_{zy}\sin\alpha_{ik}\right)du \\ &= W_{zx}dx + W_{zy}dy \end{aligned} ,$$

because

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \cos\alpha_{ik} \\ \sin\alpha_{ik} \end{pmatrix} du .$$

If points P_i and P_k are close to each other as required, integrals on the right side of equation (2) can be computed by the following trapezoidal integral approximation formula:

$$\int_i^k \left(W_{zx}dx + W_{zy}dy\right) = \int_i^k W_{zu}du \approx \frac{s_{ik}}{2}\left[\left(W_{zu}\right)_i + \left(W_{zu}\right)_k\right] \tag{3}$$

$$\int_i^k W_{zz}dz \approx \frac{\Delta h_{ik}}{2}\left[\left(W_{zz}\right)_i + \left(W_{zz}\right)_k\right] \approx \Delta h_{ik}\widetilde{W}_{zz} \approx \Delta h_{ik}U_{zz} \tag{4}$$

where s_{ik} is the horizontal distance between points P_i and P_k, Δh_{ik} is the height difference between these two points and U_{zz} is the normal value of the vertical gradient.

$$U_{zz} = \gamma\left(\frac{1}{M} + \frac{1}{N}\right) + 2\omega^2$$

where $\gamma = \gamma_e(1 + \beta\sin^2\varphi)$ is the normal gravity on the ellipsoid; *M* and *N* is the curvature radius of U in the meridian and in the prime vertical. With the values of the Geodetic Reference System 1980, the following holds at the surface of the ellipsoid:

$$U_{zz} = 3086\ ns^{-2} .$$

The value of integral (4) depends on the vertical gradient W_{zz} and the height difference between the points.

So, discarding the effect of (4) the differences of gravity values between two points can be computed by the approximate equation:

$$\begin{aligned} (g_k - g_i) \approx \frac{s_{ik}}{2}\{ & [(W_{zx})_i + (W_{zx})_k]\cos\alpha_{ik} \\ & + [(W_{zy})_i + (W_{zy})_k]\sin\alpha_{ik} \} + \Delta h_{ik}U_{zz} \end{aligned} . \tag{5}$$

If the normal value of vertical gradient U_{zz} differs significantly from the real value, taking into account the real value is necessary.

2 Practical solutions

If we have a large number of torsion balance measurements, it is possible to form an interpolation net (a simple example can be seen in Figure 2) for determining gravity at each torsion balance points (Völgyesi, 1993, 1995, 2001). On the basis of Eq. (5)

$$(g_k - g_i) = C_{ik} \tag{6}$$

can be written between any adjacent points, where

$$C_{ik} = s_{ik}\left\{ \frac{(W_{zx})_i + (W_{zx})_k}{2}\cos\alpha_{ik} + \frac{(W_{zy})_i + (W_{zy})_k}{2}\sin\alpha_{ik} \right\} . \tag{7}$$

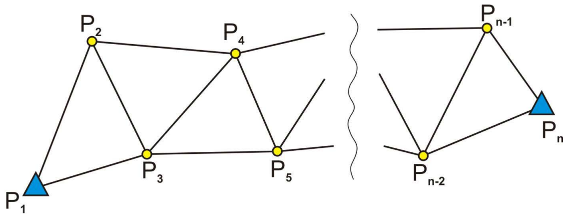

Fig. 2 Interpolation net connecting torsion balance points

For an unambiguous interpolation it is necessary to know the real gravity value at a few points of the network (triangles in Figure 2). Let us see now, how to solve interpolation for an arbitrary network with more points than needed for an unambiguous solution, where gravity values are known. In this case the g values can be determined by adjustment.

The question arises what data are to be considered as measurement results for adjustment: the real torsion balance measurements W_{zx} and W_{zy}, or C_{ik} values from Eq. (7). Since no simple functional relationship (observation equation) with a measurement result on one side and unknowns on the other side of an equation can be written, computation ought to be made under conditions of adjustment of direct measurements, rather than with measured unknowns – this is, however, excessively demanding in terms of storage capacity. Hence concerning measurements, two approximations will be applied: on the one hand, gravity values from measurements at the fixed points are left uncorrected – thus, they are input to adjustment as constraints – on the other hand, C_{ij} on the left hand side of fundamental equation (6) are considered as fictitious measurements and corrected. Thereby observation equation (6) becomes:

$$C_{ik} + v_{ik} = g_k - g_i \tag{8}$$

permitting computation under conditions given by adjusting indirect measurements between unknowns (Detrekői, 1991).

The first approximation is possible since reliability of the gravity values determined from measurements exceeds that of the interpolated values considerably. Validity of the second approximation will be reconsidered in connection with the problem of weighting.

For every triangle side of the interpolation net, observation equation (8):

$$v_{ik} = g_k - g_i - C_{ik} \tag{9}$$

may be written. In matrix form:

$$\underset{(m,1)}{\mathbf{v}} = \underset{(m,2n)}{\mathbf{A}}\,\underset{(2n,1)}{\mathbf{x}} + \underset{(m,1)}{\mathbf{l}}$$

where $\mathbf{A}$ is the coefficient matrix of observation equations, $\mathbf{x}$ is the vector containing unknowns g, $\mathbf{l}$ is the vector of constant terms, m is the number of triangle sides in the interpolation net and n is the number of points. The non-zero terms in an arbitrary row i of matrix $\mathbf{A}$ are:

$$[\dots \quad 0 \quad +1 \quad -1 \quad 0 \quad \dots]$$

while vector elements of constant term $\mathbf{l}$ are the C_{ik} values.

Gravity values fixed at given points modify the structure of observation equations. If, for instance, $g_k = g_{k0}$ is given in (8), then the corresponding row of matrix $\mathbf{A}$ is:

$$[\dots \quad 0 \quad 0 \quad -1 \quad 0 \quad \dots]$$

the changed constant term being: $C_{ij} - g_{k0}$, that is g_k, and of coefficients of g_k are missing from vector $\mathbf{x}$, and matrix $\mathbf{A}$, respectively, while corresponding terms of constant term vector $\mathbf{l}$ are changed by a value g_{k0}.

Adjustment raises also the problem of weighting. Fictitious measurements may only be applied, however, if certain conditions are met. The most important condition is the deducibility of covariance matrix of fictitious measurements from the law of error propagation, requiring, however, a relation yielding fictitious measurement results, – in the actual case, Eq. (7). Among quantities on the right-hand side of (7), torsion balance measurements W_{zx} and W_{zy} may be considered as wrong. They are about equally reliable $\pm 1E$ ($1E = 1\,Eötvös\ Unit = 10^{-9}\,s^{-2}$), furthermore, they may be considered as mutually independent quantities, thus, their weighting coefficient matrix $\mathbf{Q}_{WW}$ will be a unit matrix. With the knowledge of $\mathbf{Q}_{WW}$, the weighting coefficient matrix $\mathbf{Q}_{CC}$ of fictitious measurements C_{ik} after Detrekői (1991) is:

$$\mathbf{Q}_{CC} = \mathbf{F}^*\mathbf{Q}_{WW}\mathbf{F} = \mathbf{F}^*\mathbf{F}$$

$\mathbf{Q}_{WW} = \mathbf{E}$ being a unit matrix. Elements of an arbitrary row i of matrix $\mathbf{F}^*$ are:

$$\left[\left(\frac{\partial C_{ik}}{\partial W_{zx}}\right)_1 \quad \left(\frac{\partial C_{ik}}{\partial W_{zx}}\right)_2 \quad ,..., \quad \left(\frac{\partial C_{ik}}{\partial W_{zx}}\right)_n \right.$$
$$\left. \left(\frac{\partial C_{ik}}{\partial W_{zy}}\right)_1 \quad \left(\frac{\partial C_{ik}}{\partial W_{zy}}\right)_2 \quad ,..., \quad \left(\frac{\partial C_{ik}}{\partial W_{zy}}\right)_n \right]$$

For the following considerations let us produce rows $\mathbf{f}_1^*$ and $\mathbf{f}_2^*$ of matrix $\mathbf{F}^*$ (referring to sides between points $P_1 - P_2$ and $P_1 - P_3$ respectively):

$$\mathbf{f}_1^* = [\ \frac{s_{12}\sin\alpha_{12}}{2}, \frac{s_{12}\sin\alpha_{12}}{2}, 0, 0, ..., 0,$$
$$\frac{s_{12}\cos\alpha_{12}}{2}, \frac{s_{12}\cos\alpha_{12}}{2}, 0, 0, ..., 0\ ]$$

and

$$\mathbf{f}_1^* = [\ \frac{s_{13}\sin\alpha_{13}}{2},\ 0, \frac{s_{13}\sin\alpha_{13}}{2}, 0, 0, ..., 0,$$
$$\frac{s_{13}\cos\alpha_{13}}{2},\ 0, \frac{s_{13}\cos\alpha_{13}}{2}, 0, 0, ..., 0\ ].$$

Using $\mathbf{f}_1^*$, variance of C_{ik} value referring to side $P_1 - P_2$ is:

$$m^2 = \frac{s_{12}^2}{4}\left(2\sin^2\alpha_{12} + 2\cos^2\alpha_{12}\right) = \frac{s_{12}^2}{2}$$

while $\mathbf{f}_1^*$ and $\mathbf{f}_2^*$ yield covariance of C_{ik} values for sides $P_1 - P_2$ and $P_1 - P_3$:

$$\mathrm{cov} = \frac{s_{12}s_{13}}{4}\left(\sin\alpha_{12}\sin\alpha_{13} + \cos\alpha_{12}\cos\alpha_{13}\right).$$

Thus, fictitious measurements may be stated to be correlated, and the weighting coefficient matrix contains covariance elements at the junction point of the two sides. If needed, the weighting matrix may be produced by inverting this weighting coefficient matrix. Practically, however, two approximations are possible: either fictitious measurements C_{ij} are considered to be mutually independent, so weighting matrix is a diagonal matrix; or fictitious measurements are weighted in inverted quadratic relation to the distance.

By assuming independent measurements, the second approximation results also from inversion, since terms in the main diagonal of the weighting coefficient matrix are proportional to the square of the side lengths. The neglection is, however, justified, in addition to the simplification of computation, also by the fact that contradictions are due less to measurement errors than to functional errors of the computational model (Völgyesi, 1993).

3 Test computations

Test computations were performed in a Hungarian area extending over about 750 km^2. In the last century approximately 60000 torsion balance measurements were made mainly on the flat territories of Hungary, at present 24310 torsion balance measurements are available. Location of these 24310 torsion balance observational points and the site of the test area can be seen on Figure 3.

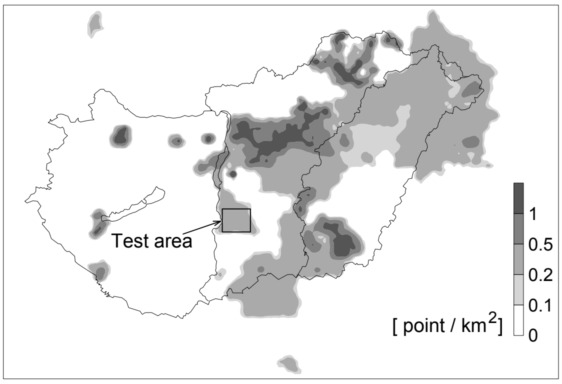

Fig. 3 Torsion balance measurements being stored in computer database, and the site of the test area

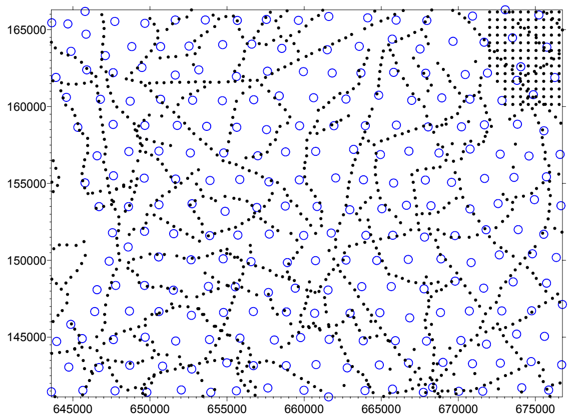

Fig. 4 Gravity measurements (marked by dots) and torsion balance points (marked by circles) on the test area

Our test area can be found nearly in the middle of Hungary (see on Fig. 3). There were 248 torsion balance stations and 1197 gravity measurements on this area. 18 points from these 248 torsion balance stations were chosen as fixed points where gravity are known from measurements and the unknown gravity values were interpolated on the remaining

230 points. Location of torsion balance stations (marked by circles) and the gravity measurements (marked by dots) can be seen on Figure 4.

Topography of the test area can be seen on fig 5, the height difference between the lowest and highest points is less than 20 m.

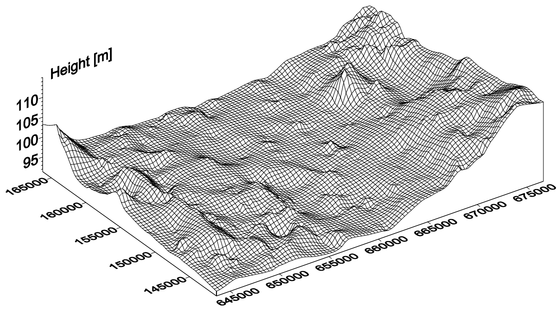

Fig. 5 Topography of the test area (heights above sea level)

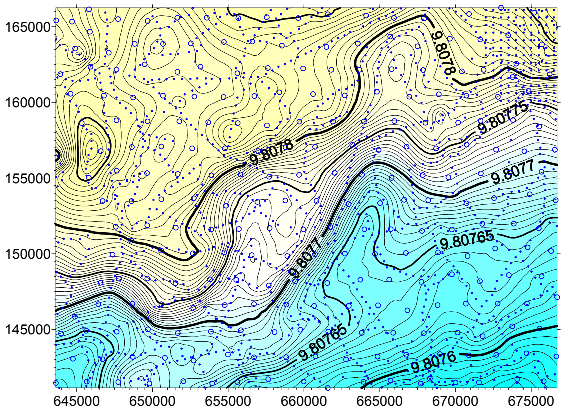

Fig. 6 Gravity field from *g* measurements on the test area

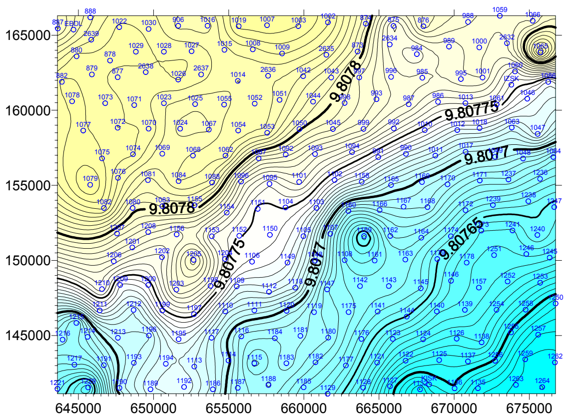

Fig. 7 Isoline map of interpolated gravity values from horizontal gradients of gravity W_{zx} and W_{zy} measured by torsion balance

The isoline map of gravity constructed from 1197 *g* measurements can be seen on Figure 6. Small dots indicate the locations of measured gravity values. Measurements were made by Worden gravimeters, by accuracy of ±20-30 μGal. At the same time the isoline map of gravity values constructed from the interpolated values from 248 torsion balance measurements can be seen on Figure 7. Small circles indicate the locations of torsion balance points.

More or less a good agreement can be seen between these two isoline maps. In order to control the applicability and accuracy of interpolation, we compared the given and the interpolated *g* values. Gravity values were determined for each torsion balance points from gravity measurements by linear interpolation on the one hand and gravity values for the same points from gravity gradients measured by torsion balance were computed on the other. Isoline and surface maps of differences between the two types of *g* values can be seen on Figures 8 and 9. The differences are about ±1−2 mGal the maximum difference is 6 mGal.

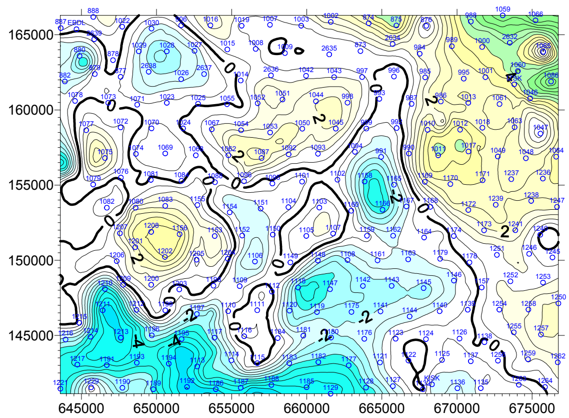

Fig. 8 Isoline map of differences between the measured and the interpolated gravity values

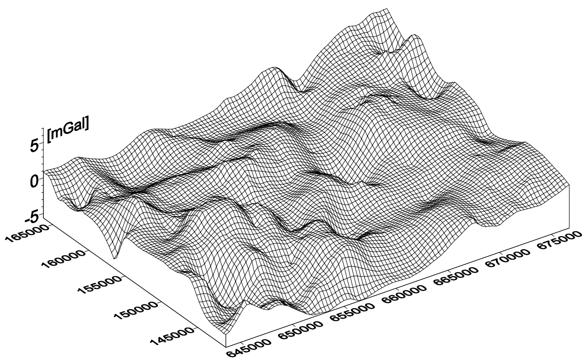

Fig. 9 Surface map of differences between the measured and the interpolated gravity values

Finally the standard error characteristic to interpolation, determined by

$$m_g = \pm\sqrt{\frac{1}{n}\sum_{i=1}^{n}(g_i^{me.} - g_i^{\text{int.}})^2} \ ,$$

was computed (where $g_i^{me.}$ are from gravity measurements, $g_i^{\text{int.}}$ are the interpolated values from torsion balance measurements and $n = 248$ is the number of torsion balance stations). Standard error $m_g = \pm 1.6$ mGal indicates that horizontal gradients of gravity give a possibility to determine gravity from torsion balance measurements by *mGal* accuracy on flat areas.

It is interesting that a correlation can be found comparing the surface map of differences between the measured and the interpolated gravity values (see Fig. 9) by the topography of the test area (see Fig. 5). The biggest errors can be found at the right side of the test area, where the biggest height differences are. In case of a not flat area accuracy of interpolation would probably be increased by taking into consideration the real vertical gradient values instead of the normal one. Unfortunately we haven't got the real vertical gradient values at torsion balance points on our test area yet.

It would be important to investigate the effect of vertical gradient's value for the interpolation in the future. The real value of W_{zz} can be computed from torsion balance measurements too, investigations are going on in this respect (Tóth-Völgyesi-Csapó, 2004).

Summary

A method was developed, based on integration of horizontal gradients of gravity W_{zx} and W_{zy}, to predict gravity values at all points of the torsion balance network. Test computations were performed in a characteristic area in Hungary where both torsion balance and gravimetric measurements are available. Comparison of the measured and the interpolated gravity values indicates that horizontal gradients of gravity give a possibility to determine gravity field from torsion balance measurements by *mGal* accuracy. Accuracy of interpolation would probably be increased by taking into consideration the real values of vertical gradient instead of normal one.

Acknowledgements

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References

Detrekői Á (1991) Adjustment calculations. Tankönyvkiadó, Budapest. (in Hungarian)

Tóth Gy, Völgyesi L, Csapó G (2004): Determination of vertical gradients from torsion balance measurements. IAG International Symposium, Gravity, Geoid and Space Missions. Porto, Portugal Aug.30 – Sep.3. (CD).

Völgyesi L (1993) *Interpolation of Deflection of the Vertical Based on Gravity Gradients*. Periodica Polytechnica Civ.Eng., Vo1. 37. Nr. 2, pp. 137-166.

Völgyesi L (1995) *Test Interpolation of Deflection of the Vertical in Hungary Based on Gravity Gradients*. Periodica Polytechnica Civ.Eng., Vo1. 39, Nr. 1, pp. 37-75.

Völgyesi L (2001) *Geodetic applications of torsion balance measurements in Hungary*. Reports on Geodesy, Warsaw University of Technology, Vol. 57, Nr. 2, pp. 203-212.

Völgyesi L, Tóth Gy, Csapó G (2004): Determination of gravity anomalies from torsion balance measurements. Gravity, Geoid and Space Missions GGSM 2004. IAG International Symposium Porto, Portugal. Jekeli C, Bastos L, Fernandes J. (Eds.) Springer Verlag, Series: IAG Symposia, Vol. 129, pp. 292-297.

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Völgyesi L, Tóth Gy (2005): Determination of gravity field from horizoltal gradients of gravity. A joint meeting of the IAG, IAPSO and IABO; Dynamic Planet 2005, Cairns, Australia, August 22-26 2005.

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